MATHEMATICAL PROBLEMS CONCERNING THE KAC MODEL

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Contents

SUMMARY

This thesis deals with the Kac model in kinetic theory. Kac's model is a linear, space homogeneous, n-particle model created by Mark Kac in 1956 in [14] in an attempt to give a derivation of Boltzmann's equation. The marginals of a distribution under Kac's evolution are connected to a simple Boltzmann-type equation via the mechanism of "propagation of chaos" given by Kac in [14].

Kac evolution preserves total (kinetic) energy and is ergodic, having the uniform distribution on the constant energy sphere as its equilibrium. The generator of the associated Markov process has a spectral gap that is bounded away from zero uniformly in n. A central question in the field is the speed of approach to equilibrium (or rate of equilibration). The thesis gives the results of the papers [25], [3], and [26] joint work with my collaborators Federico Bonetto, Michael Loss, and Ranjini Vaidyanathan.

The work in [25] extends the work in [2] by studying the rate of approach to equilibrium when a fraction $\alpha = \frac{m}{n}$ of the particles interact with a "strong" thermostat. Results in the spectral gap and the (negative of) relative entropy metric are obtained. The work in [3] shows, using both the L^2 metric and the Fourier-based metric d_2, that the evolution of the system interacting with the ideal infinite particle thermostat used in the model in [2] can be approximated by the evolution of the same system interacting with a large but finite reservoir. This approximation does not deteriorate with time, and it improves as the number of reservoir particles increases. The work in [26] studies the Kac evolution in the absence of thermostats and reservoirs using the metric d_2. It finds an upper bound to the approach to equilibrium, and constructs a family of initial states that for time t_0 independent of n shows practically no approach to equilibrium in d_2. An independent propagation of chaos result for the model in [25] is also given.

Chapter 1

Introduction

1.1 Boltzmann's Equation and Kac's Derivation Attempt

My thesis topic is the Kac model in kinetic theory. Kinetic theory is a branch of mathematical physics that studies the motion of gas particles undergoing collisions. It aims at describing macroscopic properties of a gas from microscopic properties and laws of motion. Boltzmann's equation is the central equation that is used when a gas has low-density and the collision rates are not high enough to maintain a continuous heat transfer gradients.

Boltzmann's equation describes $f(t, \vec{x}, v)$, the density of a single particle in a gas of a large number of indistinguishable particles that undergo frequent collisions. It is given by

$$\frac{\partial f}{\partial t} + v.\nabla_x f = \lambda \int_{\mathbb{R}^3} \int_{S^2} B\left(\sigma.\frac{v-v^*}{|v-v^*|}, |v-v^*|\right) |v-v^*| \left(f(v^{*\prime})f(v') - f(v^*)f(v)\right) d\sigma dv^*, \tag{1.1}$$

where x belongs to a region Ω, there is no external forcing term in the equation, $\sigma \in S^2$, and $v^{*\prime}$ and v' are $\frac{1}{2}(v+v^*) \pm \frac{1}{2}|v-v^*|\,\sigma$ respectively.

$$\int_{x\in\Omega} \int_{v\in\mathbb{R}^3} f(t,\vec{x},v)\, d\vec{x}\, dv = 1 \qquad \text{for all times.}$$

The $v.\nabla_x f$ is a transport term, coming from the straight line motion of the particles when they are not colliding. The right-hand side is the collision term. It is nonlinear in f and the integral is in only the velocity variables. For integrable B, this collision term consists of a gain term (the terms with $f(v^{*\prime})f(v')$) and a loss term (the term with $f(v^*)f(v)$).

This equation is a good approximation when the particle density is small and the particles are at high temperature. These conditions minimize the chance of more than two particles colliding at a time, and minimize the effect of interparticle interactions when no collisions are taking place. The interparticle interactions are summarized in the collision term/kernel $B(.,.)$. If the interparticle forces are derived from a potential, ϕ, one can derive an expression for the scattering angle (and thus B) using the conservation of energy and momentum (see [12]). The computation is similar to that of Rutherford scattering.

Maxwell discovered that if ϕ is a power law $\phi \propto r^{-(s-1)}$, with $s > 3$ then

$$B(., |v|) \propto |v|^{-\frac{4}{s-1}}.$$

So choosing $s = 5$, the collision term becomes independent of $|v - v^*|$ (see chapter 3 in [12]). Particles interacting with this potential are known as Maxwellian molecules. Using Maxwellian molecules in the context of the Kac-Boltzmann equation will simplify our computations in later sections.

Boltzmann's equation qualitatively tells us about approach to equilibrium. Let $f(t, \vec{x}, v)$ be a solution to Boltzmann's equation and let $H[f]$ be given by

$$H[f] = \int f \ln(f) d\vec{x} dv. \tag{1.2}$$

Then, Boltzmann showed that H decreases with time with $\frac{dH}{dt} = 0$ only if f is a local Maxwellian $A(x)e^{-B(x)(v-C(x))^2}$, with special relations among the $A, B, C(x)$. f is a global Maxwellian, except for some special initial conditions which allow for non-constant A, B, and C. Desvillettes and Villani made this convergence quantitative for smooth global solutions.

This equilibration property of solutions to Boltzmann's equation seems to violate Poincaré's recurrence theorem for Hamiltonian systems, which says that on a compact space, a volume preserving dynamical system eventually returns arbitrarily closely to its initial state. Poincaré's theorem applies for our system since space is confined to Ω and the total energy $\sum_i v_i^2 = nE$ is conserved. This precludes that the Boltmann equation approximates the Hamiltonian evolution for all times. The size of the recurrence time, however, is physically irrelevant $(10^{10\times \text{ age of the universe}})$ (units in seconds. see [12] section 5.3 and the references given there.)

Boltzmann's equation has not been rigorously derived from first principles, that is, from a many body Hamiltonian system. Lanford [15, 16] proved the validity of Boltzmann's equation only for times that are of the order of the mean free time between collisions. One way to obtain it is to start from Liouville's equation using the n particle density function $f_n(t, x_1, \ldots, v_n)$ and Boltzmann's molecular chaos assumption.

The molecular chaos assumption, which is also known as Boltzmanns Stosszahlansatz, is essential in the derivation of Boltzmann's equation. It says that prior to a collision between two particles, their velocities can be taken to be independent. That is, if $f_i(x_1, \ldots, x_i, v_1, v_2, \ldots, v_i)$ is the i-particle marginal of the velocity distribution of the particles, then

$$f_2(t, x_1, x_2, v_1, v_2) = f_1(t, x_1, v_1) f_1(t, x_2, v_2) \tag{1.3}$$

whenever the particles (x_1, v_1) and (x_2, v_2) have not collided yet.

Liouville's equation reads:

$$\frac{\partial f_n}{\partial t} + \sum_{i=1}^{n} \vec{v_i}.\nabla_{\vec{x_i}} f_n + \frac{1}{m} \sum_{i=1}^{n} \vec{F_i}.\nabla_{v_i} f_n = 0 \tag{1.4}$$

with $\vec{F_i}$ the total force acting on particle i. We take the force to be due to pair interactions that result from a potential $\phi_{i,j} = \phi(|v_i - v_j|)$. If the particles are modelled as hard spheres of radius r, we can take the external forces $\vec{F_i}$ to be zero when all the particles are at a distance of more than $2r$ away.

Fixing $(x_1, \ldots, x_k, v_1, \ldots, v_k)$ and integrating (1.4) over the variables $(x_{k+1}, \ldots, x_n, v_{k+1}, \ldots, v_n)$ on the region where $(x_1, \ldots, x_n) \in \Omega$ and $\min\{|x_i - x_j| : 1 \leq i < j \leq n\} \geq 2r$, and invoking the divergence theorem gives an equation relating f_k to f_{k+1}. This hierarchy of equations is known as the BBGKY hierarchy.

Boltzmann's theorem now results in Grad's limit: $r \to 0$ and $n \to \infty$ in such a way that nr^2 $\to$constant, using Boltzmann's Stosszahlansatz. We notice that the density, proportional to nr^3, goes to zero. This also allows the forbidden region $(|x_i - x_j| < 2r)$ in the integro-differential equation for f_1 to go to zero. It is interesting to note that for hard spheres $\pi r^2 \bar{v}$ is proportional to the average volume of a region travelled by a particle per unit time. Here $\bar{v}$ is the typical speed of the particle. So the average number of collisions per particle per unit time is $cst \times \frac{n}{V} r^2 \bar{v}$. Hence, the typical size of the mean free path is proportional to $(nr^2)^{-1}$; since $\bar{v}/$(mean number of collisions per unit time) is proportional to $(nr^2)^{-1}$.

Kac [14] created a stochastic model in an attempt to give a rigorous derivation of the Boltzmann equation. Although his derivation is not from Hamiltonian mechanics, it is a sound probabilistic derivation which clarifies the role of molecular chaos. Kac made some simplifications in his model which are presented next. The density of the particles is spatially homogeneous, and thus independent of x. Even for the Boltzmann equation, being able to solve for the spatially homogenous case first is an important step since the particle motion can be divided into a succession of streamings and collisions. Kac also assumed that his particles are Maxwellian. This simplification makes the scattering cross section $B(.,v)|v|$ independent of the momentum transfer.

Kac also took the particles to move in 1-dimension. In this case, the conservation of both momentum and energy requires that during a collision particles either pass through each other keep their own velocities or they exchange their velocities. Since this would be too restrictive, Kac gave up the conservation of angular momentum in 1-dimension. As momentum conservation is connected with pressure which, in a spatially homogeneous case, is not relevant, conservation of energy becomes more important since energy has temperature as its intensive counterpart, which is relevant in the spatially homogeneous case. The last two simplifications are not that restrictive since passing from 1D to 3D was possible for many of the results obtained, as in [4] and [6].

1.2 Kac's Model and Propagation of Chaos

In Kac's model, there are n indistinguishable particles with mass equal to 1 moving in 1 dimension. The dynamical variable is $(v_1, v_2, \ldots, v_n)$, the one-dimensional velocites of the particles. The particles undergo Kac collisions as follows. At a collection of times $\{t_i\}_{i=1}^{\infty}$, which are separated exponentially at rate r with the $t_i - t_{i-1}$ independent, 2 particles (j_1, j_2) are chosen uniformly and at random to collide. Before giving the details of the collision, we first choose the rate r in such a way that the rate of collision of each particle is independent of the total number of particles. Let's fix particle i_0. The probability that particle i_0 is involved in a collision given that a collision has taken place is:

$$\frac{n-1}{\binom{n}{2}} = \frac{2}{n}.$$

After time 1, there are on average r collisions. Thus particle i_0 collides on average about $\frac{2r}{n}$ times during 1 second. Since we are interested in the limit of infinite n, we set

$$r = n\lambda$$

for some λ independent of n. This makes particle i_0 collide on average 2λ times per second, independently of the total number of particles.

Kac assumed that in the collision, the velocities (v_{j_1}, v_{j_2}) get replaced by $(v^*_{j_1}(\alpha), v^*_{j_2}(\alpha))$ where

$$\begin{cases} v^*_{j_1}(\alpha) & = v_{j_1}\cos\alpha + v_{j_2}\sin\alpha \\ v^*_{j_2}(\alpha) & = v_{j_1}\sin\alpha - v_{j_2}\cos\alpha \end{cases},$$

and α is chosen from a distribution $\rho(\alpha)$ that satisfies $\rho(2\pi-\alpha) = \rho(\alpha)$. This property of ρ is called local reversibility because it makes the probability of going from (v_{j_1}, v_{j_2}) to $(v^*_{j_1}, v^*_{j_2})$ equal to the probability of going from $(v^*_{j_1}, v^*_{j_2})$ to (v_{j_1}, v_{j_2}). We will take $\rho(\alpha) \equiv \frac{1}{2\pi}$ throughout the rest of this thesis.

This collision preserves the total energy [1] $\sum_{i=1}^n v_i^2 =: nE$. Here E is the average kinetic energy per particle. So if $f(t, v_1, v_2, \ldots, v_n)$ is the velocity density at time t, we can take f to be supported on a sphere $\sum v_i^2 = nE$ if $f(0, v_1, v_2, \ldots, v_n)$ is supported on the same sphere.

The effect of the (j_1, j_2) collision on f is represented by $Q_{j_1,j_2}\phi$ with

$$Q_{j_1,j_2}[\phi] = \fint_0^{2\pi} d\theta\phi(\ldots, v_{j_1}\cos\theta - v_{j_2}\sin\theta, \ldots, v_{j_1}\sin\theta + v_{j_2}\cos\theta, \ldots). \tag{1.5}$$

The Kac collision operator is given by Q with

$$Q[f](v_1, \ldots, v_n) = \binom{n}{2}^{-1} \sum_{i<j} Q_{i,j}f = \binom{n}{2}^{-1} \sum_{i<j} \fint_0^{2\pi} f(\ldots, v^*_i, \ldots, v^*_j, \ldots)d\theta. \tag{1.6}$$

Let $p_k(t)$ denote the probability that ,by time t, k collisions have taken place. Then $p_k = e^{-n\lambda t}\frac{(n\lambda t)^k}{k!}$, the law of the Poisson distribution with mean $n\lambda t$. Thus if the initial distribution is f_0, then the distribution at time t is the following weighted sum.

[1] Take the mass m to be 2.

$$\begin{aligned} f(t,\dots) &= \sum_{k=0}^{\infty} p_k Q^k[f] = e^{-n\lambda t} f_0 + e^{-n\lambda t}(n\lambda t) Q f_0 + \dots + e^{-n\lambda t}\frac{(n\lambda t)^k Q^k}{k!} f_0 + \dots \\ &= e^{n\lambda t(Q-I)} f_0. \end{aligned}$$

So $f(t,\dots)$ satisfies the Kolmogoroff forward equation (or Fokker-Planck equation)

$$\frac{\partial f}{\partial t} = n\lambda(Q-I)f, \tag{1.7}$$

known as the Kac master equation.

Integrating both sides on $\mathbb{R}^n$ we see that the master equation preserves $\int f(v)\,dv$. Since the particles are indistinguishable, we will take the initial condition f_0 to be symmetric in its variables (untill we break this symmetry by thermostating some of the particles in chapter 2). This symmetry property is preserved under the Kac evolution.

Note that Q is positivity preserving because $Q[f]$ is a convex combination of the $f(Q_{i,j}(\theta)[v])$ for various i, j and θ. Here

$$Q_{i,j}(\theta)[(v_1,\dots,v_n)] = (v_1,\dots,v_{i-1}, v_i\cos\theta - v_j\sin\theta,\dots,v_{j-1}, v_i\sin\theta + v_j\cos\theta, v_{j+1},\dots,v_n). \tag{1.8}$$

For the same reason, Q is an averaging operator: $Q[1](v) \equiv 1$ and

$$\int_{S^{n-1}(\sqrt{nE})} Q[f](v)\,\sigma^n(dv) = \int_{S^{n-1}(\sqrt{nE})} f(v)\,\sigma^n(dv),$$

for all $f \in L^1\left(S^{n-1}(\sqrt{nE})\right)$. Here σ^n is the normalized uniform probability measure on the sphere S^{n-1}.

Kac found the following link between the n-particle Kac master equation and the one-particle Kac-Boltzmann equation (1.9).

$$\frac{\partial \phi}{\partial t}(v,t) = \lambda \fint_0^{2\pi} \int_{\mathbb{R}} \phi(v\cos\theta - w\sin\theta, t).\phi(v\sin\theta + w\cos\theta, t) - \phi(v,t)\phi(w,t)\,dw\,d\theta. \tag{1.9}$$

Suppose $\{f_n : S^{n-1}(\sqrt{nE}) \to [0,\infty) \in L^1(symm)\}_n$ is a sequence of probability densities relative to the normalized uniform measures on spheres, with the property that for some $\phi \in$

$L^1(\mathbb{R})$ and all integers k

$$\lim_{n\to\infty}\int_{S^{n-1}(\sqrt{nE})} f_n(v)h(v_1,v_2,\ldots,v_k)\sigma^n(dv) = \int_{\mathbb{R}^k}\prod_{i=1}^{k}\phi(v_i)\,h(v)\,dv \qquad (1.10)$$

for all h bounded and continuous. Kac showed that

$$\lim_{n\to\infty}\int_{S^{n-1}} e^{nt\lambda(Q_n-I)}[f_n](v)h(v_1,v_2,\ldots,v_k)\sigma^n(dv) = \int_{\mathbb{R}^k}\prod_{i=1}^{k}\phi(v_i,t)\,h(v)\,dv,$$

where $\phi(v,t)$ solves the Kac Boltzmann equation (1.9).

In other words, if the sequence $\{f_n\}_n$ is chaotic (to ϕ), then this chaotic property holds for future times under the Kac evolution and the chaotic limit solves a Boltzmann-type equation referred to as the Kac-Boltzmann equation. Kac called this property the propagation of the Boltzmann property. This property is also known as propagation of chaos.

One can think of chaoticity as a weak form of independence. The velocities of the n-particles are not independent if they are restricted on a sphere. But if in the limit of infinite n, any fixed k of them become independent, then this property will hold in future times, even though collisions introduce correlations among the particle velocities. Propagation of chaos was used to prove the uniqueness of the solution to the Boltzmann-Kac equation [20, 24]. We will show a propagation of chaos result for a related model in chapter 2.

A first example of a chaotic sequence is $\{f_n : S^{n-1}(\sqrt{nE}) \mapsto [0,\infty), f_n(v)\equiv 1\}$. Computations similar to (6.5) in Appendix B give

$$\int_{S^{n-1}(\sqrt{nE})} h(v_1,\ldots,v_k)\sigma^n(dv) = \frac{|S^{n-k}|}{|S^{n-1}|}\int_{B^k(0,\sqrt{nE})} h(v_1,\ldots,v_k)\left(1-\frac{|v|^2}{nE}\right)^{\frac{n-k}{2}} dv_1\ldots dv_k,$$

which converges to $(2\pi E)^{-\frac{k}{2}}\int_{\mathbb{R}^k} h(v_1,\ldots,v_k)e^{-\frac{v^2}{2E}}\,dv$ as n approaches infinity. So $\{f_n\}$ is chaotic to the centered Gaussian distribution with average energy E. This example shows that with larger n, any k of the variables v_i on the sphere $\sum_{i=1}^n v_i^2 = nE$ become less and less dependent. One reason this is interesting is because it can be seen as a central limit theorem, and as a passage from the microcanonical ensemble to the canonical ensemble in statistical mechanics. Another reason is that propagation of chaos can be quickly verified for this sequence. The constant function 1 is a stationary state for the Kac model on $S^{n-1}(\sqrt{nE})$. Similarly, the

Gaussians are stationary states for Kac-Boltzmann equation (1.9). So chaoticity property holds for all times and therefore chaos propagates.

One way to show the existence of solutions to the Kac-Boltzmann equation with initial condition $f(0,v) = f_0(v)$ is to show that there is a family $\{f_n : S^{n-1}(\sqrt{nE}) \to [0,\infty) \in L^1(symm)\}_n$ of probability densities that is chaotic with limit f_0. This raises the question of which probability distributions on $\mathbb{R}$ are limits of a chaotic sequence of distributions on spheres. This question, which is also interesting in its own right, was studied by Carlen and his coauthors in [5]. They showed that if f_0 satisfies

$$\int f_0(v)v^4\,dv < \infty, \quad f_0 \in L^p \quad \text{for some } p \text{ in } (1,\infty) \tag{1.11}$$

then f_0 is a limit of a chaotic sequence. Note that in [5] a stronger version of propagation of chaos is proven. For our purposes it suffices to mention that the restriction of $\prod_{i=1}^n f_0(v_i)$ on the sphere $\sum_{i=1}^k v_i^2 = kE$ is meaningful in the sense of L^1 if f_0 satisfies conditions (1.11).

1.3 Approach to Equilibrium and the Spectral Gap

Kac used the space $L^2(S^{n-1}(\sqrt{nE}),\sigma^n)$ because it is convenient to deal with from the function-analytic point of view. We will show in Lemma 1 of chapter 2 that if the initial distribution $f(0,v)$ is a function in L^2, then $f(t,v)$, the solution of (1.7), is also a function in L^2. We will also see, in the same lemma, that the Kac collision operator is ergodic on spheres. That is, the constant function is the only invariant distribution in $L^2(S^{n-1}(\sqrt{nE}),\sigma^n)$ and the operator $-L$ in equation (1.7) is strictly negative as a quadratic form on the space of L^2 functions which are orthogonal to the constrants (See [4]). Fundamental questions in the field include: How fast is the approach to equilibrium in the Kac model? Can we choose a metric and a family of physically interesting initial conditions where the approach to equilibrium is of order 1 or order $\ln(n)$? Or do we have to wait for time of order n?

A lot of work has been done in the above direction. We now look at some of the results that are relevant to this work. Kac conjectured that his operator $n(I-Q)$ has a spectral gap Δ_n in $L^2(S^{n-1}(\sqrt{nE}),\sigma^n)$ independent of n. That is, if

$$\Delta_n = \inf\left\{\int_{S^{n-1}} f(v)[n(I-Q)f](v)\sigma^n(dv) : \int_{S^{n-1}} f(v)\sigma^n(dv) = 0, \text{and} \int_{S^{n-1}} f(v)^2\sigma^n(dv) = 1\right\}, \tag{1.12}$$

then $\Delta_n \geq c > 0$ with c independent of n.

Since the constant function 1 is in the kernel of $n(I-Q)$, and $n(I-Q)$ is positive definite, the spectral gap provides a rate of approach to equilibrium. In fact, if the initial distribution $f(0,.) \in L^2(S^{n-1}(\sqrt{nE}), \sigma^n)$ then the distribution at time t satisfies $||f(t,.)-1||_{L^2} \leq e^{-\Delta_n t}\,||f(0,.)-1||_{L^2}$.

From Lemma 1, it is easy to show that $\Delta_n \geq 0$. It is not obvious why $\Delta_n > 0$ even for finite n. Nevertheless, this conjecture was proven by Janvresse in [13]. The exact gap, for more general models, was computed by Carlen, Carvalho, and Loss [4]. This gap for the regular Kac model with $\lambda = 1$ is λ_1:

$$\lambda_1 = \frac{n+2}{2(n-1)}. \tag{1.13}$$

This result implies that if the initial distribution $f(0,.)$ is a function in $L^2(S^{n-1}(\sqrt{nE}), \sigma^n)$ then the distribution at time t, under the evolution (1.7) satisfies

$$||f(t,.)-1||_{L^2} \leq e^{-\lambda_1 t}\,||f(0,.)-1||_{L^2} \leq e^{-\frac{t}{2}}\,||f(0,.)-1||_{L^2}$$

for any n. This result proves Kac's conjecture, but does not show convergence in L^1 in time of order 1. This is because the the bound $||f(t,.)-1||_{L^1(S^{n-1}(\sqrt{nE}),\sigma^n)} \leq ||f(t,.)-1||_{L^2(S^{n-1}(\sqrt{nE}),\sigma^n)}$ is very crude and there are "chaotic" probability densities on $\mathbb{R}^n$ restricted to the sphere $S(\sqrt{nE})$ with L^2 norm proportional to C^n for some $C > 1$.

In fact, if we let $\phi(x)$ be the function $\mathbf{1}_{[0,\infty)}(x)e^{-x^2}$ in $L^1(\mathbb{R})$ and let $\psi_n(v)$ be the probability density in $L^1(S^{n-1}(\sqrt{nE}), \sigma^n)$ given by

$$\psi_n(v) = \prod_{i=1}^{n} \phi(v_i) \Big/ \int_{S^{n-1}(\sqrt{nE})} \prod_{i=1}^{n} \phi(w_i)\sigma^n(dw),$$

then $\psi_n(v)$ corresponds to the probability density $2^n\mathbf{1}_{\{w:w_i\geq 0 \forall i\}}(.)$ in $L^1 \cap L^2(S^{n-1}(\sqrt{nE}), \sigma^n)$. We know that $||\psi_n - 1||_{L^2}^2 = 2^{2n} - 1$.

This shows that we require time of order n to guarantee decay in L^1 even for chaotic initial distributions. For this reason, the relative entropy distance $S(f|1)$, an extensive quantity similar to Boltzmann's H functional, was studied to circumvent this problem. Here 1 signifies the uniform distribution on the sphere and the relative entropy is given by the equation

$$S(f|1) = \int_{S^{n-1}(\sqrt{nE})} f(v) \ln \frac{f(v)}{1} \, \sigma^n(dv).$$

In proposition 3 we will show that $S(f|1)$ serves as a distance between f and the uniform measure on the sphere. Under the Kac evolution $S(f(t)|1) \to 0$ as $t \to \infty$. A simple but non-quantitative proof is given in Proposition 5. Villani gave a quantitative upper bound in [21]. He showed that $S(e^{n\lambda t(Q-I)} f|1) \leq S(f_0|1)e^{-2\lambda t/(n-1)}$ by proving that the initial entropy production, $-\frac{1}{S(f(t)|1)} \frac{d}{dt} S(f(t)|1)\Big|_{t=0+}$, is at least $\frac{2}{n-1}$ for all initial distributions f which are not indentically 1 and that have finite relative entropy. Einav showed in [10] that Villani's entropy production result is essentially exact at $t = 0$ by constructing states whose initial entropy production can be practically as small as $\frac{C}{n}$.

Decay rates in S immediately result in decay rates in L^1 because of the Kullback, Leibler, Czizar, and Pinsker inequality: (see McKean [20] sec. 9 for a nice proof)

$$||f - 1||_{L^1}^2 \leq 2S(f|1). \tag{1.14}$$

In chapter 2, where revisit S and give some of its properties, together with a proof of the the limit $\lim_{t\to\infty} S(f(t)|1) \to 0$.

The states constructed by Einav gave half the total energy of the system to a vanishingly small fraction of the particles. This is not physically probable and raises the question: which initial states should we consider as physically acceptable? Bonetto, Loss, and Vaidyanathan argued in [2] that a good candidate for a physical initial condition is one close to equilibrium, so all the $\mathcal{N}$ particles except n (with $1 << n << \mathcal{N}$) are independent of the rest of the particles, and have Gaussian distribution $\Gamma_{n,\beta}$ at a fixed inverse temperature β.

$$\Gamma_{n,\beta}(v) = \left(\frac{\beta}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} e^{-\frac{\beta}{2}v_i^2}.$$

They constructed a model with n particles that undergo Kac collisions and that also interact with an infinite heat bath (the thermostat) at inverse temperature β (as an approximation to the $\mathcal{N}$-particle reservoir). The action of the Maxwellian thermostat on the i^{th} particle is given by

$$M_i[f](v_1, \ldots, v_n) = \int_{w\in\mathbb{R}} \fint_{\theta=0}^{2\pi} f(v_1, \ldots, v_i \cos\theta - w\sin\theta, \ldots, v_n)\Gamma_{1,\beta}(v_i \sin\theta + w\cos\theta)\, d\theta\, dw \tag{1.15}$$

Here, particle i undergoes Kac collision with a thermostat particle that has velocity w coming from the Gaussian distribution $\Gamma_{1,\beta}$. We integrate out the variable w, thus removing this thermostat particle from the picture. The master equation for this thermostated Kac model is

$$\frac{\partial f}{\partial t} = n\lambda(Q - I)f + \mu \sum_{i=1}^{n} (M_i - I)f =: -L_T(f) \tag{1.16}$$

with μ the rate of the thermostat. Since the thermostat can pump energy in or out of the system, the underlying space is now $\mathbb{R}^n$ instead of $S^{n-1}(\sqrt{nE})$ and $f \in L^1(\mathbb{R}^n)$. Similarly, $S(f|1)$ has to be replaced by $S(f|\Gamma_{n,\beta})$, measuring distance towards the equilibrium.

They showed that the first spectral gap in L^2 for L_T is $\frac{\mu}{2}$ which is bounded below away from zero independently of n. Since this only depends on the rate of the thermostat μ and not on λ, the rate of the Kac collisions, the also obtained the second gap. The second gap is attained by 4^{th} order polynomials and depends on both μ and λ. They also showed that the entropy production for the evolution via (1.16) is at least $\frac{\mu}{2}$. This implies decay in entropy in time of order 1 which, together with inequality (1.14), shows order 1 convergence in L^1 too.

1.4 What's New?

My first work is [25], joint with Ranjini Vaidyanathan. The question we studied is what happens if we thermostat only a fraction m/n of the particles? The master Equation (1.16) has to be replaced by the following:

$$\frac{\partial f}{\partial t} = n\lambda(Q - I)f + \mu \sum_{i=1}^{m} (M_i - I)f. \tag{1.17}$$

Part of our motivation was the fact that the first L^2 gap and the decay rate of $S(f|\Gamma_{n,\beta})$ in [2] are independent of λ and are valid even when $\lambda = 0$. This overshadows the role of the Kac collisions in equilibration. By only thermostating m of the n particles, the Kac collisions between particles interacting with the thermostat and particles not interacting with the thermostat becomes necessary in taking the distribution of the whole system to the Gaussian equilibrium state.

We used a thermostat which is "stronger" than the Maxwellian thermostat in [2], and took our master equation to be

$$\frac{\partial f}{\partial t} = -L_{n,m} f = n\lambda(Q - I)f + \mu \sum_{i=1}^{m} (P_i - I)f, \tag{1.18}$$

with P_i the operator for the strong thermostat acting on particle i. It is given by

$$P_i[f](v_1, \ldots, v_n) = \Gamma_{1,\beta}(v_i) \int_{\mathbb{R}} f(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n)\, dw. \tag{1.19}$$

When the thermostat in this model acts on a particle, the particle forgets its precollisional velocity and picks a new velocity from the Gaussian distribution at the same temperature as the thermostat.

We obtained quantitative rates on the approach of $f(t, v)$ to $\Gamma_{n,\beta}(v)$ and showed using an inductive argument in $m \leq n - 1$ and n that $L_{m,n}$ in (1.18) has a spectral gap $\Delta_{n,m} = O(\frac{m}{n})$. This result is summarized in Theorem 2 below. The relative entropy was much more difficult to study. We showed that $S(f(t)|\Gamma_{n,\beta}) \to 0$ essentially exponentially, at least at a rate of order $\frac{m}{n^2}$, which we think is suboptimal in its m/n behavior. This result is summarized in Theorem 3 below.

Finally, we showed that the "strong" thermostat and the Maxwellian thermostat are related. The Maxwellian thermostat can be obtained from the strong thermostat as a Van Hove limit (weak coupling, large time limit. See [9]). These results are given in Theorem 4 in chapter 2. A propagation of chaos result is true for the partially thermostated Kac model (see [26]). It requires taking n and m to be a integer multiples of integers n_0 and m_0, and leads to a system of Kac-Boltzmann equations. m_0/n_0 is the ratio of particles thermostated. This is given in Theorem 5 below.

My second work is [3], joint with Bonetto, Loss, and Vaidyanathan. We looked at the validity of the infinite thermostat assumption in the model in [2]. Consider a large set of n-system particles and $\mathcal{N} >> n$ reservoir particles. Take the initial condition to be

$$f(v, w) = l_0(v)\Gamma_{\mathcal{N},\beta}(w).$$

We allow the system and reservoir particles to undergo both Kac collisions among their respective groups and mixed Kac collisions, and allow system-reservoir collisions at a rate chosen in such a way that on average a system particle collides a comparable amount of times with reservoir particles and with other system particles. The generator of the evolution is

$(-L_{FR})$ given by equation (4.5) in chapter 4. The minus sign makes L_{FR} positive definite in $L^2(\mathbb{R}^{n+\mathcal{N}}, \Gamma_{n+\mathcal{N}})$.

This process needs to be compared with the evolution via the infinite thermostat $-L_T$ in (1.16). To allow comparisons, we let L_T act on functions of $n+\mathcal{N}$ variables by leaving the last $\mathcal{N}$ variables intact. The assumption in [2] can be made rigorous by putting it in the form: "Both evolutions

$$e^{-tL_{FR}}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)], \qquad e^{-tL_T}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$$

stay close to each other as long as the n-system particle marginals are concerned; the difference in marginals tends to zero with large $\mathcal{N}$ and fixed n, uniformly in t."

We justified a stronger form of this assumption by showing both in the L^2-metric and, under a technical 4^{th} moment assumption, in the Fourier based GTW metric d_2 that the two evolutions stay close for all times t, and the difference in their evolution goes to zero as $\mathcal{N} \to \infty$. Not only are the n-particle marginals of these evolutions close, but also the whole distributions. Our convergence rates are uniform in time and are summarized in Theorems 6 and 7 in chapter 4.

My Third work [26], is motivated by the successful application of the GTW metric ,d_2, in [3] and in other works [22, 7, 23]. I studied the Kac master equation, equation (1.7), using d_2. The object of study here is

$$d_2(e^{-tL}\mu, R_\mu)$$

where μ is a Borel probability measure on $\mathbb{R}^n$ that has zero mean and finite energy, e^{-tL} is the Kac evolution operator of equation (1.7) that can easily be adapted to measures. Here R_μ denotes the angular average of μ, the infinite time limit of $e^{-tL}[\mu]$. If A is a measurable set, then $R_\mu(A)$ is given by

$$R_\mu(A) = \int_{Q\in SO(n)} \mu\left(Q^{-1}[A]\right)\mathcal{H}(dQ). \tag{1.20}$$

Here $\mathcal{H}$ is the normalized Haar measure on $SO(n)$. We will show in chapter 3 (eq. (3.4)) that $d_2(e^{-tL}\mu, R_\mu)$ is nonincreasing in time.

The physically interesting case is when μ corresponds to indistinguishable particles. So we take μ to be invariant under the exchange of the coordinates in $\mathbb{R}^n$.

Two results are proven in this direction. The first is given by Theorem 8 which shows that essentially we have

$$d_2(e^{-tL}\mu, R_\mu) \leq K_\mu e^{-\frac{4\lambda\lambda_1}{n+3}t}$$

where K_μ is a constant that only depends of $\int v_1^2\mu(dv)$ and $\left|\int v_1v_2\mu(dv)\right|$, and λ_1 is the gap of (1.13), and λ is proportional single particle collision rate.

The second result is given by Theorem 9. We construct functions f_n that for time up to $\frac{1}{2\lambda}$, $\frac{d_2(e^{-tL}f_n, R_{f_n})}{d_2(f_n, R_{f_n})}$ is effectively 1. An interesting feature is that this bound survives for $t = O(1)$, unlike Einav's lower bound on entropy production in [10] which is known to be valid only in an infinitesimal interval at $t = 0$.

The d_2 metric is based on the Fourier transform. It was introduced by Gabetta, Toscani, and Wennberg in the context of the Kac Boltzmann equation in [22], and is sometimes called the GTW-metric d_2. It was noticed in [3] that d_2 is intensive when it is comparing chaotic distributions on $\mathbb{R}^n$, that is, distributions whose variables are all independent of each other. We will see in chapter 3 that this intensivity property can be generalized to non-chaotic distributions provided one allows these distributions to undergo the Kac evolution in (1.7) for time order $\ln(n)$ (See [26]).

The thesis is organized as follows. We give the results of the paper [25] and the propagation of chaos result in [26] in chapter 2, together with precise related statements and notation from [2]. We provide properties of the d_2 metric in chapter 3. We give the results of the paper [3] in chapter 4. In chapter 5 we give the convergence in d_2 result and present the functions with slow d_2 decay constucted in [26]. We summarize in chapter 6 the results and pose some open questions. Appendices $A1$ and $A2$ give us the spectrum of $L_{2,1}$ and a sequence of functions with large values of $S(Q_{1,2}f|\Gamma_2) + S(P_1[f]|\Gamma_2)$. Appendix B gives us the L^2 and d_2 distances between the initial system reservoir state $l_0(v)\Gamma_{\mathcal{N}}(w)$ and its radial projection. We see in Appendix C the optimality of Lemma 6 in the $\frac{1}{\mathcal{N}}$ behavior. Finally, we see in Appendix D why the upper bound in (4.30) is not linear in $d_2(l_0, \Gamma_n)$.

1.5 Notation

1. $b_{2i} = \fint \cos(\theta)^{2i}\, d\theta$ are the nonzero eigenvalues of the Maxwellian thermostat M_i (see equation (2.5)); $b_{2i,2j} = \fint (\cos\theta)^{2i}(\sin\theta)^{2j}\, d\theta$.
2. $B^k(c, r)$ is the ball of center c and radius r in $\mathbb{R}^d$.
3. "chaotic distribution": a probability density (or measure) f on $\mathbb{R}^k$ is chaotic if its variables are independent (See for example (3.5)). It is different from "chaotic sequence" or "a sequence being chaotic to" a limit (See (3.5)).
4. d_2 the Gabetta-Toscani-Wennberg metric (see (3.1)).
5. Δ_n is the spectral gap of the Kac operator L (See (1.12)).
6. $\Delta_{n,m}$ is the spectral gap of the partially thermostated Kac operator $\bar{L}_{n,m}$ (See (2.8)).
7. E is the average enrgy per particle, an intensive quantity (See the paragraph before (1.7)).
8. $E_{4,k}, E_{\phi,k}$ denote moments of ϕ_k . $E_{4,k} = \int v_1^4 \phi_k(v)\, dv$. (See Lemma 3).
9. $\hat{f}(\xi)$ or $\mathcal{F}[f](\xi)$. Is $\int_{\mathbb{R}^k} f(v) e^{-2\pi i v.\xi}\, dv$. The Fourier transform of f.
10. $\Gamma_{k,\beta}(v) = \left(\frac{\beta}{2\pi}\right)^{\frac{k}{2}} e^{-\frac{\beta}{2}\sum_{i=1}^k v_i^2}$. Sometimes, denoted by Γ_k. $\Gamma(x)$ stands for Euler's gamma function in Appendix B. Γ is a derivation on the algebra $Z(\mathbb{R}^\infty,\ symm)$ in section 2.4.
11. $H_k(x)$ is the monic Hermite polynomial of degree k with weight $\Gamma_{1,\beta}(x)$. (See (2.4)).
12. I is the identity operator, $I[f] = f$. As a superscript, I stands for "interaction", as in $Q^I_{i,j}$.
13. Intensive. A function $q(n)$ is intensive in n if $q(n)$ is independent of n at least for large enough n. $q(n)$ is extensive if it is roughly linear in n. (See Proposition 7).
14. $\fint_0^{2\pi} \phi(\theta)\, d\theta = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \phi(\theta)\, d\theta$.
15. L or $\mathcal{L}$ denotes a generic positive definite operator. Could stand for the Kac collision term, with or without the thermostats. e.g. $n\lambda(I - Q)$.
16. $L_{n,m} = n\lambda(I - Q) + \mu\sum_{j=1}^m (I - P_j)$ (See (1.18)).
17. $\bar{L}_{n,m} = n\lambda(I - Q) + \mu\sum_{j=1}^m (I - \bar{P}_j)$ (See (2.7)).

18. Λ is a positive constant like $n\lambda$ depending on L that makes $(\Lambda I - L)/\Lambda$ an averaging operator (See for example (4.15)).

19. $L_{FR} = \mathcal{L}_k + \mathcal{L}_I$ evolution for the FR-system (see (4.5)).

20. $L_T = \mathcal{L}_k + \mathcal{L}_T$ is the evolution for the T-system (see (4.6)).

21. $M_i =$ Maxwellian thermostat acting on the i^{th} particle (see (1.15)).

22. $\bar{M}_i =$ Maxwellian thermostat acting on the i^{th} particle in the h-representation (see (2.3)).

23. μ can be a number, the rate of the thermostat OR a Borel measure, giving the initial velocity distribution of the particles.

24. n stands for number of particles in the system (not reservoir or thermostat).

25. m, $m \leq n$ is the number of thermostated particles. $\frac{m}{n} =: \alpha = O(1)$.

26. $\mathcal{N}$ is the number of reservoir particles, $n << \mathcal{N}$.

27. $P_i =$ strong thermostat acting on the i^{th} particle (P for projection) in the f world (see (1.19)).

28. $\bar{P}_i =$ same as P_i but in the h world (see (2.6)).

29. $Q_{i,j}$ Kac rotation operator (see (1.5)). $Q_{i,j}[\theta]$ is the isometry on $\mathbb{R}^n$ that affects the i and j component of a vector. (See (1.8) and (3.10)).

30. $Q^I_{i,j}$ interaction term and $m\mu Q^I = \frac{\mu}{n}\sum_{i=1}^m \sum_{j=1}^n Q^{(I)}_{i,j}$ (See (4.4) and (4.3)).

31. $\mu\, n\, Q_M = \mu \sum_{i=1}^n M_i$, is the Maxwellian thermostat acting on the system particles.

32. $Q_{\bar{M}}$ is the Maxwellian thermostat operator Q_M in the h-world. (See (4.12)).

33. $Q^z_{x,y}$ is always a positivity preserving averaging operator, no matter what subscripts or superscripts it is given (See (4.1)-(4.3)).

34. R could stand for a radial function or the reservoir in the model in Chapter 4.

35. R_h denotes the radial projection (angular average) of h: $R_h(v) = \int_{S^{n-1}(|v|)} h(w)\sigma^r(dw)$. R_μ can be defined similarly for Borel probability measures μ. Note that taking the radial projection and taking the Fourier transform commute.

36. σ^r denote the uniform probability measure on $S^{n-1}(r)$ for and $r > 0$, Sometimes denoted by σ or σ^n (See the equations before and after (1.9)). σ also denotes a permutation in section 2.4.

37. $|h|^p_{L^p(r)}$ is $\int_{S^{n-1}(r)} |h(w)|^p \sigma^r(dw)$ for $1 \leq p < \infty$, whenever h is supported on the sphere $S^{(n-1)}(r)$.

38. $|h|_{L^\infty(r)} =$ *ess sup*$\{|h(w)| : |w| = r\}$; whenever h is supported on the sphere $S^{(n-1)}(r)$ (See the paragraph after (5.2)).

39. $\langle h_1, h_2 \rangle = \int_{\mathbb{R}^k} h_1(v)\, h_2(v) \Gamma_k(v)\, dv$, whenever $h_1, h_2 \in L^2(\mathbb{R}^k, \Gamma_k)$.

40. $S(f|h) = \int f \ln(f/h)\, dv$ is negative of the relative entropy of f with respect to h. Sometimes denoted $S(f)$ if $h = 1$ or h is clear from the context; sometimes $S(\phi)$ denotes $S(\phi\Gamma_n|\Gamma_n)$.

41. symmetric distribution. A symmetric distribution on $\mathbb{R}^n$ (or S^{n-1}) is a distribution f that is invariant under any permutation of the variables $v_1, \ldots, v_n$. $L^1(symm)$ denotes the symmetric functions in L^1. (See the paragraph after (1.9)).

42. $v, w, \vec{\xi}, \vec{\eta}$: $v \in \mathbb{R}^n$ contains the velocities of the n system particles and $\vec{\xi}$ contains their Fourier frequencies . $w \in \mathbb{R}^{\mathcal{N}}$ contains the velocities of the reservoir particles, $\vec{\eta}$ containes the corresponding Fourier frequencies (see chapter 5, and section 5.1 for the Fourier variables).

43. $\wedge$: $a \wedge b = \min\{a, b\}$. $\hat{f}$ stands for the Fourier transform of f. On a variable $\hat{v}_i$ means v_i is to be removed: $dv_1 \ldots \widehat{dv_i} \ldots dv_n = \prod_{j \neq i} dv_j$.

44. $\vee$: $a \vee b$ stands for the maximum of a and b. $a_+ = \max\{a, 0\}$.

45. $\vec{\eta}^j$ or η^j, the vector in $\mathbb{R}^{k-1}$ obtained from $\vec{\eta} \in \mathbb{R}^k$ by removing its j^{th} coordinate (See (4.25)).

Chapter 2

Fully and Partially Thermostated Kac Models

In this chapter, we take as starting point the fully thermostated Kac model, present the function spaces and operators used, and give the related convergence results about the spectral gap and the relative entropy proven in [2]. Then we make the connection with the partially thermostated Kac model and prove the analogous results.

2.1 Maxwellian Thermostats, $L^2(\mathbb{R}^n, \Gamma_{n,\beta})$, Relative Entropy

In the model constructed in [2], n system particles undergo Kac collisions at a rate λ and interact with particles from an infinite thermostat at temperature $\frac{1}{\beta}$ at a rate μ. Before the collision the particles in the thermostat are assumed to be independent of the system particles and have distribution $\Gamma_{1,\beta}(v) = \sqrt{\frac{\beta}{2\pi}} e^{-\frac{\beta}{2}v^2}$. The effect of a collision between system particle i and a thermostat particle on the distribution is represented by the operator M_i given by (1.15) which we restate:

$$M_i[f](v_1, \dots, v_n) = \int_{w \in \mathbb{R}} \fint_{\theta=0}^{2\pi} f(\dots, v_i \cos\theta - w \sin\theta, \dots) \Gamma_{1,\beta}(v_i \sin\theta + w \cos\theta)\, d\theta\, dw.$$

Here only the i^{th} argument of f is affected by the Maxwellian thermostat and the dots indicate the variables $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{n-1}$ and v_n.

After a collision, this model assumes that the thermostat particle never collides again and can be ignored. When the number of particles in the thermostat is finite, this assumption is not valid. In this case, as we show in chapter 4, the evolution of the system plus reservoir in this model will stay close to the evolution of the same system plus (finite) reservoir under Kac collisions with conveniently chosen rates. In chapter 4 we show that the difference goes to zero as the population of the reservoir grows indefinitely. The master equation for this model is equation (1.16):

$$\frac{\partial f}{\partial t} = n\lambda(Q-I)f + \mu\sum_{i=1}^{n}(M_i - I)f =: -L_T(f).$$

Here we take f from $L^1(\mathbb{R}^n)$ instead of $L^1(S^{n-1}(\sqrt{nE}), \sigma^n)$ since the thermostat can change the energy of the system.

Notice that the Gaussian distribution $\Gamma_{n,\beta}(v)$ is invariant under the action of the thermostats M_i. Since $\Gamma_{n,\beta}(v)$ is also a radial function, it is invariant under Kac collisions Q too. Thus $\Gamma_{n,\beta}(v)$ is an equilibrium distribution of equation (1.16).

It is important to note that $\Gamma_{n,\beta}$ is the only stationary distribution to (1.16) in L^1. The proof of this claim gives the opportunity to introduce some of the properties of the Kac-collision and the Maxwellian thermostat operators, and an auxiliary L^2 space that will be used multiple times.

Consider the transformation $f(v) = h(v)\Gamma_{n,\beta}(v)$. We will first take h to be a function in $L^2(\mathbb{R}^n, \Gamma_n)$.

Lemma 1 *As a quadratic form on $L^2(\mathbb{R}^n, \Gamma_n)$, $0 \le Q \le I$. The same is true on $L^2(S^{n-1}, \sigma)$.*

Proof.- Since Q is a convex combination of the Kac rotations $Q_{i,j}$, and each of the $Q_{i,j}$ is a projection into the space of functions that depend on v_i and v_j only through $\sqrt{v_i^2 + v_j^2}$, we have $Q_{i,j} \ge 0$. So Q is a sum of nonnegative quadratic forms, and is thus nonnegative.

To show $Q \leq I$, we consider:

$$
\begin{aligned}
\int |h - Q_{i,j}h|^2 \Gamma(v) dv &= \int h(v)^2\, \Gamma dv - 2\int h(v) Q_{i,j}[h](v)\Gamma(v)\, dv + \int |Q_{i,j}h|^2\, \Gamma(v) dv \\
&= \int h(v)^2 \Gamma(v)\, dv + \int |Q_{i,j}h\,|^2\, \Gamma(v) dv \\
&-2\int\fint_0^{2\pi} [hQ_{i,j}h]\,(v_i^*, v_j^*)\, d\theta \sqrt{v_i^2+v_j^2}\Gamma_2(\sqrt{v_i^2+v_j^2}) d\sqrt{v_i^2+v_j^2}\,\Gamma_{n-2}(v_1, \hat{v}_i, \hat{v}_j, v_n) \prod_{k=\neq i,j} dv_k \\
&= \int h(v)^2 \Gamma(v)\, dv + \int |Q_{i,j}h\,|^2\, \Gamma(v) dv \qquad (2.1)\\
&-2\int Q_{i,j}h(v) \left(\fint_0^{2\pi} h(v_i^*, v_j^*)\, d\theta\right) \sqrt{v_i^2+v_j^2}\Gamma_2(\sqrt{v_i^2+v_j^2}) d\sqrt{v_i^2+v_j^2}\,\Gamma_{n-2}(v_1, \hat{v}_i, \hat{v}_j, v_n) \prod_{k\neq i,j} dv_k \\
&= \int h(v)^2\, \Gamma(v) dv + \int [Q_{i,j}h]^2\, \Gamma(v)\, dv - 2\int (Q_{i,j}h)^2\, \Gamma(v)\, dv \\
&= \int h(v)^2 \Gamma(v) dv - \int (Q_{i,j}h)^2\, \Gamma(v)\, dv \geq 0.
\end{aligned}
\tag{2.2}
$$

Here, in equation (2.1), we used the fact that $Q_{i,j}h(v)$ depends on v_i and v_j only via $\sqrt{v_i^2 + v_j^2}$ and not on the angle θ between them. And we used the fact that $\left(\fint_0^{2\pi} h(v_i^*, v_j^*)\, d\theta\right) = Q_{i,j}[h](v)$. This identity shows that $||Q_{i,j}h||_{L^2} \leq ||h||_{L^2}$ with equality only if $Q_{i,j}h = h$ almost everywhere.

The result with $Q_{i,j}$ replaced by Q follows from the inequalities $||Qh||_{L^2} \leq \binom{n}{2}^{-1} \sum_{i<j} ||Q_{i,j}h||_{L^2}$ which is less than or equal to $||h||_{L^2}$ with equality if and only if h is a constant a.e. $\square$

As a corollary, we have that if $f(0, v) \in L^2$, then $f(t, v)$, the solution to the master equation (1.7), belongs to L^2 since $f(t, v)$ is the combination of the $Q^k f$ with absolutely summable weights p_k (see the equation before (1.7)).
The actions M_i on f induce the actions $\bar{M}_i$ on h given by

$$M_i[h(v)\Gamma_{n,\beta}(v)] = \bar{M}_i[h](v)\Gamma_{n,\beta}(v),$$

where

$$\bar{M}_i[h](v) = \int_{\mathbb{R}}\fint_0^{2\pi} h(v\cos\theta - w\sin\theta)\Gamma_{n,\beta}(w) d\theta\, dw. \tag{2.3}$$

The following lemma is proven in [2].

Lemma 2 *$\bar{M}_1$ is self-adjoint on $L^2(\mathbb{R}, \Gamma_{1,\beta})$, and $0 \leq \bar{M}_1 \leq I$ as quadratic forms. Furthermore $\int h\bar{M}_1[h]\, \Gamma(v)\, dv = \int |h|^2 \Gamma(v)\, dv$ if and only if $h = cst$ almost everywhere.*

Proof.- The symmetry of $\bar{M}_1$ follows from the following

$$\begin{aligned}\langle \bar{M}_1 h_1, h_2\rangle_{\Gamma_1} &= \int_{\mathbb{R}}\int_{\mathbb{R}}⨍_0^{2\pi} h_1(x\cos\theta - y\sin\theta)h_2(x)\Gamma(x)\Gamma(y)\,d\theta\,dx\,dy \\ &= ⨍_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} h_1(x^*(\theta))h_2(x)\Gamma(x^*(\theta))\Gamma(y^*(\theta))\ dx^*(\theta)\,dy^*(\theta)\,d\theta,\end{aligned}$$

where $(x*(\theta), y^*(\theta)) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$ as in the introduction. This transformation has Jacobian 1, and $x = x^*(\theta)\cos\theta + y^*(\theta)\sin\theta$, and we have used the identity $x^*(\theta)^2 + y^*(\theta)^2 = x^2 + y^2$ in going from $\Gamma(x)\Gamma(y)$ to $\Gamma(x^*(\theta))\Gamma(y^*(\theta))$

$$\begin{aligned}\langle \bar{M}_1 h_1, h_2\rangle_{\Gamma_1} &= ⨍_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} h_1(x^*(\theta))h_2(x^*(\theta)\cos\theta + y^*(\theta)\sin\theta)\Gamma(x^*(\theta))\Gamma(y^*(\theta))\ dx^*(\theta)\,dy^*(\theta)\,d\theta \\ &= ⨍_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} h_1(x^*(\theta))h_2(x'\cos\theta + y'\sin\theta)\Gamma(x')\Gamma(y')\ dx'\,dy'\,d\theta \\ &= \langle h_1, \bar{M}_1 h_2\rangle_{\Gamma_1}.\end{aligned}$$

The self-adjointness of $\bar{M}_1$ follows from the fact that its domain is the whole of $L^2(\mathbb{R}, \Gamma_{1,\beta})$. The second claim follows from the following proposition which gives the spectrum of $\bar{M}_1$. $\square$

Let H_k be the monic Hermite polynomial of degree k with weight $\Gamma_{1,\beta}$. So

$$H_k(x) = \frac{(-1)^k}{\beta^k} e^{\beta x^2/2}\frac{d^k}{dx^k}e^{-\beta x^2/2}, \tag{2.4}$$

and we have $\int_{\mathbb{R}} H_k(v)H_l(v)\Gamma_1(v)\,dv = 0$ when $k \neq l$. We have

Proposition 1

$$\bar{M}[H_k] = b_k H_k(v) \tag{2.5}$$

where $b_k = ⨍_0^{2\pi}(\cos\theta)^k\,d\theta$.

This proposition was proven in [2]. We give the proof here for convenience.

Proof.- To obtain them we notice that if $p(v)$ is a polynomial of degree k, $\bar{M}[p](v)$ is a polynomial of degree less than or equal to k. Thus, if $k < l$, then

$$\int \bar{M}[H_k](v)H_l(v)\Gamma(v)\,dv = 0,$$

and if $l < k$, by the self-adjointness of $\bar{M}$, we have

$$\int \bar{M}[H_k](v)H_l(v)\Gamma(v)\,dv = \int H_k(v)\bar{M}[H_l](v)\Gamma(v)\,dv = 0,$$

Thus, $\bar{M}[H_k] = c_k H_k$.

To compute the eigenvalues c_k, we examine the action of $\bar{M}$ on v^k.

$$\bar{M}[v^k] = \int (v\cos\theta - w\sin\theta)^k \Gamma(w)\,dwd\theta = b_k \int (v^2+w^2)^{\frac{k}{2}}\Gamma(w)\,dw.$$

$b_k = 0$ when k is odd. And for k even, the right-hand side is $b_k v^k$ plus terms of lower order in v. This shows that $c_k = b_k$. □

In particular, we have $\bar{M}_i \le I$ as quadratic forms, and we have

$$\int_{\mathbb{R}^1} h(v)\bar{M}_1[h](v)\Gamma_{1,\beta}(v)\,dv = \int_{\mathbb{R}^n} h(v)^2\Gamma_{1,\beta}(v)\,dv$$

if and only if $h = $ cst a.e.

Now we are ready to prove the following therorem.

Theorem 1 *$\Gamma_{n,\beta}$ is the only equilibrium to eq.* (1.16) *in* $L^1(\mathbb{R}^n)$.

In the proof we first show that $h=1$ is the only function in $L^2(\Gamma_{n,\beta})$ such that $h\Gamma_{n,\beta}$ is stationary under eq. (1.16), and then, using a density argument and applying LaSalle's principle, we show that $h=1$ is the only function in $L^1(\Gamma_{n,\beta})$ such that $h\Gamma_{n,\beta}$ is stationary under eq. (1.16).

Proof.- Let $f = h\Gamma_n$ with $h \in L^2(\mathbb{R}^n, \Gamma_n)$. Consider $\int_{\mathbb{R}^n} h(t,v)^2\Gamma_{n,\beta}(v)\,dv$.
Then $\frac{d}{dt}\left(\int_{\mathbb{R}^n} h(t,v)^2\Gamma_{n,\beta}(v)\,dv\right) < 0$ because

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathbb{R}^n} h(t,v)^2\Gamma_{n,\beta}(v)\,dv\right) &= \int_{\mathbb{R}^n} 2h(t,v)[n\lambda(Q-I)[h](t,v)]\Gamma_{n,\beta}(v)\,dv \\ &= (2n\lambda)\left\{\int_{\mathbb{R}^n} h(t,v)Q[h](t,v)\Gamma_{n,\beta}(v)\,\Gamma_n(v)dv - \int_{\mathbb{R}^n} h(t,v)^2\,\Gamma_{n,\beta}(v)\,dv\right\} \\ &\le 0\end{aligned}$$

by Lemma 1 with equality only when h is a constant. Since $\int f(v)\,dv = 1$, we must have $h \equiv 1$ and $f(v) = \Gamma_{n,\beta}(v)$.

To prove the result for any f in $L^1(\mathbb{R}^n)$, we need to pass from $h \in L^2(\mathbb{R}^n, \Gamma_n)$ to $h \in L^1(\mathbb{R}^n, \Gamma_n)$. This is a standard 3ϵ arguement. Let h be a probability density in $L^1(\mathbb{R}^n, \Gamma_n)$.

Consider $h \wedge M_\epsilon$, where $M_\epsilon \geq 1$ is chosen large enough so that $\int h \wedge M_\epsilon \Gamma_n(v)\, dv \geq 1 - \epsilon$. The thermostated evolution (1.16) induces an evolution $e^{-t\bar{L}}$ on h which preserves the L^1 norms of nonnegative functions. Hence $\int_{\mathbb{R}^n} e^{-t\bar{L}}[h \wedge M_\epsilon]\Gamma_n(dv) \geq 1 - \epsilon$ for all t. $h \wedge M_\epsilon \in L^1 \cap L^2(\Gamma_n)$ and thus $e^{-t\bar{L}}[h \wedge M_\epsilon]$ converges in L^2 to $(\int h \wedge M_\epsilon \Gamma_n) \times 1$. So

$$e^{-t\bar{L}}[h \wedge M_\epsilon] \to (\int_{\mathbb{R}^n} h \wedge M_\epsilon \Gamma_n(v) dv) \text{ a. e.}$$

and $||e^{-t\bar{L}}[h \wedge M_\epsilon]||_{L^1(\Gamma_n)} \to ||(\int_{\mathbb{R}^n} h \wedge M_\epsilon \Gamma_n(v)\, dv)||_{L^1(\Gamma_n)}$. Thus, by a corollary of Fatou's lemma (see Theorem 1.9 in [18]) we have $e^{-t\bar{L}}[h \wedge M_\epsilon] \to \int h \wedge M_\epsilon \Gamma_n(v)\, dv$ in L^1. Therefore

$$\begin{aligned}
\limsup_{t\to\infty} ||e^{-tL}h - 1||_{L^1(\Gamma_n)} \;\leq\; & \limsup_{t\to\infty} ||e^{-tL}h - e^{-tL}[h \wedge M_\epsilon]||_{L^1(\Gamma_n)} \\
& + \limsup_{t\to\infty} ||e^{-tL}[h \wedge M_\epsilon] - \int h \wedge M_\epsilon \Gamma_n(v)\, dv||_{L^1(\Gamma_n)} \\
& + \limsup_{t\to\infty} \left|\left| \int (h \wedge M_\epsilon - 1)\Gamma_n(v)\, dv \right|\right|_{L^1(\Gamma_n)} \\
& \leq 2\epsilon + \limsup_{t\to\infty} ||e^{-tL}[h \wedge M_\epsilon] - \int h \wedge M_\epsilon \Gamma_n(v)\, dv||_{L^1(\Gamma_n)} = 2\epsilon.
\end{aligned}$$

Since ϵ was arbitrary, $e^{-tL}h \to 1$ is $L^1(\Gamma_n)$, which implies that $e^{-tL}f \to \Gamma_n$ in $L^1(\mathbb{R}^n, dv)$. $\square$

The first L^2-gap for this model is defined by:

$$\Delta_n = \inf\left\{ \langle \phi, (n\lambda(I - Q) + \mu \sum_{i=1}^{n} (\bar{M}_i - I))\phi \rangle : \int \phi \Gamma_n\, dv = 0, \int \phi^2 \Gamma_n\, dv = 1 \right\}.$$

Here we can think of ϕ as $h - 1$. In [2] it is shown that $\Delta_n = \frac{\mu}{2}$ with $\phi = \sum_{i=1}^{n} \left(v_i^2 - \frac{1}{\beta} \right)$.

A beautiful result in the fully thermostated Kac model is Proposition 2.10 in [2] which played a key role in the proof of the convergence estimate of the relative entropy. We restate it here in Proposition 2. Its proof uses a clever change of variables which brings in the Ornstein-Uhlenbeck process and exploits the Log-Sobolev inequality for the Ornstein-Uhlenbeck process.

Proposition 2 *Let $h \geq 0$ be a probability density in $L^1(\mathbb{R}, \Gamma_{1,\beta})$ and $\bar{M}$ be the Maxwellian thermostat given by* (2.3). *Then*

$$S(\bar{M}[h]\Gamma_{1,\beta}|\Gamma_{1,\beta}) \leq \frac{1}{2} S(h|\Gamma_{1,\beta}).$$

This proposition, together with the convexity property of entropy, implies that $S(e^{-tL_T} f|\Gamma_n) \leq e^{-\frac{\mu}{2}t} S(f|\Gamma_n)$ with L_T as in (1.16). This result remains true even if the rate of the Kac collision is zero. The thermostat alone is sufficient to drive the system to the Gaussian-equilibrium state.

We now look at some of the properties of the relative entropy. First we show that $S(f|\phi)$ can serve as a distance between probability measures.

Proposition 3 *Let μ be a measure on $\mathbb{R}^n$ and f and ϕ be probability densities on $L^1(\mathbb{R}^n, \mu)$ with $S(f\mu|\phi\mu) < \infty$. Then $\int f \ln \frac{f}{\phi}\, d\mu \geq 0$, with equality if and only if $f = \phi$ μ a.e.*

This follows from $\int f \ln \frac{f}{\phi}\, \mu(dv) = \int \frac{f}{\phi} \ln \frac{f}{\phi} \phi\, \mu(dv) \geq \left(\int f \mu(dv) \right) \ln \left(\int f \mu(dv) \right) = 0$, where Jensen's inequality was used together with the fact that $\int f(v)\, \mu(dv) = 1$. Because $x \ln x$ is strictly convex, equality to zero in the above implies that $f = \phi\ \mu - a.e.$ □

We now give in Proposition 4 a convexity property of the relative entropy, which again follows from convexity of the function $x \ln x$.

Proposition 4 *Let f_i, h be probabily densities on $\mathbb{R}^n$ for $i \leq N$ and let $\sum_{i=1}^N c_i = 1$ with $c_i \geq 0$. Let $S(f_i|h) < \infty$ for each i. Then*

$$S\left(\sum_{i=1}^N c_i\, f_i \middle| h \right) \leq \sum_{i=1}^N c_i S(f_i|h).$$

Here N can be infinite. We will use this inequality with $c_k = e^{-n\lambda t} \frac{(n\lambda t)^k}{k!}$, the probability of k collisions in time t, and $h = \Gamma_{n,\beta}$.

The following proposition relates S to the Kac evolution and is analogous to Boltzmann's H-theorem.

Proposition 5 *Let f_0 be a probability distribution on $\mathbb{R}^n$ with radial projection R_f, and let $f(t)$ denote $e^{-tn\lambda(I-Q)} f_0$, the evolution of f_0 after Kac collisions. Then $S(f(t)|R_f)$ is decreasing, and $S(f(t)|R_f) \to 0$ as $t \to \infty$.*

To prove this proposition, we note that by Jensen's inequality, $S(Qf|R_f) \le S(f|R_f)$ with equality if and only if $Qf = f$ a.e. or equivalently, f is constant a.e. on every sphere. This follows from fact that $Q_{i,j}$ is an averaging operator, $x \ln x$ is convex, and that $Q_{i,j}R_f = R_f$ with equality if and only if $Q_{i,j}[f] = f$ a.e.

We have that $S\left(e^{-\lambda tn(Q-I)}f \middle| R_f\right)$ is decreasing in t since if $t_2, t_1 \ge 0$, we have:

$$S(e^{-\lambda(t_1+t_2)n(Q-I)}f|R_f) \le \sum_{k=0}^{\infty} e^{-n\lambda t_2}\frac{(n\lambda t_2)^k}{k!}S(Q^k e^{-\lambda t_1 n(Q-I)}f|R_f) \le S(e^{-\lambda t_1 n(Q-I)}f|R_f),$$

with equality if and only if $e^{-\lambda t_1 n(Q-I)}f$ is constant on each sphere centered at the origin. Notice that this happens only when $f(t=0)$ is constant on each sphere centered at the origin because of the invertibility of $e^{-\lambda t_1 n(Q-I)}$ and the fact that $e^{-\lambda t_1 n(Q-I)}1 \equiv 1$ on each sphere. Thus, $S(f(t)|R_f)$ is strictly decreasing and $S \ge 0$.

We now show $\lim_{t\to\infty} S(f(t)|R_f) = 0$.

Let $X = L^1(\mathbb{R}^n)$, $D_E = \{f \in X : f \ge 0, \int f = 1$, and $\int f(\ln\frac{f}{R_f})_+ \, dv \le E\}$. Here x_+ denotes $\max\{x, 0\}$. By Fatou's lemma, D is closed for every E, and if f_0 is a probability density on $\mathbb{R}^n$ with $\int f(\ln\frac{f}{R_f})dv = E_0$, then: $f(t) \in D_{E_0+\frac{1}{e}}$ for all t [1] Thus, by an infinite dimensional version of LaSalle's stability principle [17], to show that $S(f(t)|R_f) \to 0$ it remains to show that $S(f(t)|R_f)$ is a Lyapunoff function in the sense that $\limsup_{t\to 0+}(S(f(t+h)|R_f) - S(f(t)|R_f)/h \le 0$, with equality only if $f = R_f$ a.e. This follows from

$$\frac{d}{dt}S(f(t)|R_f) = n\lambda\left(\int_{S^{n-1}(nE)} Q[f]\ln(f) - f\ln(\frac{f}{R_f})dv\right),$$

and the observation

$$\begin{aligned}
\int Q_{i,j}[f]\ln\frac{f}{R_f}dv &= \int Q_{i,j}[f]Q_{i,j}[\ln\frac{f}{R_f}]dv \\
&\le \int Q_{i,j}[f]\ln[\frac{Q_{i,j}f}{R_f}]dv \\
&\le \int f\ln\left[\frac{f}{R_f}\right]dv.
\end{aligned}$$

[1] For $|x\ln x| \le e^{-1}$ when $x \in (0,1]$.Thus $\int f(t)\left[\ln\frac{f}{R_f}\right]_+ dv = \int f(t)\left[\ln\frac{f}{R_f}\right] + \int f(t)\left[\ln\frac{f}{R_f}\right]_-$ which is less than or equal to $E_0 + \int_{\{|f(t)/R_f|\le 1\}}\frac{f(t)}{R_f}\left|\ln[\frac{f(t)}{R_f}]R_f(v)\,dv\right| \le E_0 + \frac{1}{e}$.

2.2 Partially Thermostated Kac Model

We now introduce the partially thermostated Kac model. In [25] we used a stronger thermostat. The action of this strong thermostat on the i^{th} particle is represented by the operator P_i, and is given by equation (1.19) which we restate:

$$P_i[f](v_1, \dots, v_n) = \Gamma_{1,\beta}(v_i) \int_{\mathbb{R}} f(v_1, \dots, v_{i-1}, w, v_{i+1}, \dots, v_n)\, dw.$$

As mentioned in the introduction, after the strong thermostat acts on a particle, this particle forgets its velocity and picks a new velocity according to a Gaussian at the same temperature as the thermostat. One can think of this strong thermostat as the result of multiple actions of the Maxwellian thermostat on the particle, after which the velocity of the particle gradually becomes independent of its initial velocity, and its velocity distribution becomes Gaussian. The strong thermostat does this in 1 step. Another way to think about the action of the strong thermostat is to switch the system particle with a thermostat particle.

In the partially thermostated Kac model we take the master equation to be equation (1.18):

$$\frac{\partial f}{\partial t} = -L_{n,m} f = n\lambda(Q - I)f + \mu \sum_{i=1}^{m} (P_i - I) f$$

with $m < n$. We are interested in the case where $\frac{m}{n} \approx \alpha$ as $n \to \infty$. $L_{n,m}$ preserves the symmetry among the first m variables and the symmetry in the last $n - m$ variables. So we take the initial condition $f(0, .)$ to be symmetric in its first m variables, and in its last $n - m$ variables and $f(t, .)$ will have the same symmetry.

The Kac term $n(Q-I)$ alone takes a function to its radial projection (angular average), while the thermostat term alone takes a function towards $\Gamma_{m,\beta}(v_1, \dots, v_m) f_{m+1,n}(v_{m+1}, \dots, v_n)$. Here $f_{m+1,n}(v) = \int_{\mathbb{R}^m} f(w, v_{m+1}, \dots, v_n)\, dw$. So the Gaussian $\Gamma_{n,\beta}$, being both radial and having $\Gamma_{m,\beta}$ as its marginal is an equilibrium for this equation. We study the spectral gap of $L_{n,m}$ in the auxiliary space $h \in L^2(\Gamma_n)$, after making the transformation $f(v) = (1 + h(v))\Gamma_{n,\beta}(v)$. The action of the thermostat in this transformation is represented by $\bar{P}_i$ which is given by

$$P_i[f] = (1 + \bar{P}_i[h])\Gamma_{n,\beta}(v).$$

and

$$\bar{P}_i[h](v) = \int_{\mathbb{R}} \int_0^{2\pi} h(v_1, \ldots, v_i \cos\theta - w \sin\theta) \Gamma_{1,\beta}(w) \, d\theta dw. \tag{2.6}$$

The operator $\bar{P}_i$ is self-adjoint, and so it is a projection. We will write

$$\frac{\partial h}{\partial t} = -\bar{L}_{n,m} h = n\lambda(Q - I)h + \mu \sum_{i=1}^{m} (\bar{P}_i - I)h \tag{2.7}$$

Similar to the result in the fully thermostated Kac model, 1 is the only equilibrium to (2.7) in $L^2(\mathbb{R}^n, \Gamma_n)$ and in $L^1(\mathbb{R}^n, \Gamma_n)$. So $\Gamma_{n,\beta}$ is the only equilibrium solution to (1.18) in $L^1(\mathbb{R}^n, 1dv)$.

Approach to Equilibrium in L^2

Let $\Delta_{n,m}$ denote the spectral gap of this model. It depends on λ and μ, but we will suppress them in the notation.

$$\Delta_{n,m} = \inf \left\{ \langle h, \bar{L}_{n,m}[h] \rangle : \int_{\mathbb{R}^n} h(v) \Gamma_{n,\beta}(v) \, dv = 0, \int_{\mathbb{R}^n} h(v)^2 \Gamma_{n,\beta}(v) \, dv = 1 \right\}. \tag{2.8}$$

When $n = 2$ and $m = 1$, $Q = Q_{1,2}$ becomes a projection to the radial functions and thus, the spectrum of $\bar{L}_{2,1}$ is easy to compute. This is done in Appendix A1. The gap in this case is the smaller root of the quadratic equation $x^2 - (2\lambda + \mu)x + \lambda\mu = 0$. Namely:

$$\Delta_{2,1} = \frac{(2\lambda + \mu) - \sqrt{4\lambda^2 + \mu^2}}{2} \tag{2.9}$$

with gap eigenfunction

$$\frac{2\lambda}{2\lambda + \mu - \Delta_{2,1}} H_2(v_1) + \frac{2\lambda}{2\lambda - \Delta_{2,1}} H_2(v_2)$$

given in terms of the monic Hermite polynomials.

The next theorem shows that $\Delta_{n,m}$ is of the order $\frac{m}{n}$. The proof is by induction in n which is similar in spirit to the induction used in the computation of the L^2 gap in [4].

Theorem 2 *Assume $\lambda, \mu > 0$. Then*

$$\frac{m}{n-1} \Delta_{2,1} \leq \Delta_{n,m} \leq \frac{m}{n-1} \frac{2\lambda\mu}{\mu + \lambda} \,. \tag{2.10}$$

The proof is by induction. We first prove the following claim for $1 \leq m < n$:

$$\Delta_{n,m} \geq \frac{n-m-1}{n-1}\Delta_{n-1,m} + \frac{m}{n-1}\Delta_{n-1,m-1} \ . \tag{2.11}$$

We let $\bar{L}^{(k)}_{n,m}$ be the evolution operator $\bar{L}_{n,m}$ with the k^{th} particle removed:

$$\bar{L}^{(k)}_{n,m} = \frac{(n-1)\lambda}{\binom{n-1}{2}} \sum_{\substack{i<j \\ i,j\neq k}}^{n} (I - Q_{ij}) + \mu \sum_{\substack{l=1 \\ l\neq k}}^{m} (I - \bar{P}_l).$$

Remark 1 *$\bar{L}^{(k)}_{n,m}$ is also self-adjoint in $L^2(\mathbb{R}^n, \Gamma_n)$, and will have m or $m-1$ thermostats in it, depending on whether $k > m$ or $k \leq m$, respectively. The Kac collision term in $\bar{L}^{(k)}_{n,m}$ is among $n-1$ particles.*

Note that

$$\bar{L}_{n,m} = \frac{1}{n-1}\sum_{k=1}^{n} \bar{L}^{(k)}_{n,m}. \tag{2.12}$$

because

$$\begin{aligned}
\sum_{k=1}^{n} \bar{L}^{(k)}_{n,m} &= \sum_{k=1}^{n} \left(\frac{2\lambda}{n-2} \sum_{\substack{i<j, \\ i,j\neq k}}^{n} (I - Q_{ij}) + \mu \sum_{\substack{l=1 \\ l\neq k}}^{m} (I - \bar{P}_l) \right) \\
&= 2\lambda \sum_{i<j}^{n} (I - Q_{ij}) + (n-1)\mu \sum_{l=1}^{m} (I - \bar{P}_l) \\
&= (n-1)\bar{L}_{n,m}.
\end{aligned}$$

Thus

$$\langle h, \bar{L}_{n,m}[h]\rangle = \frac{1}{n-1}\sum_{k=1}^{n} \langle h, \bar{L}^{(k)}_{n,m}[h]\rangle \tag{2.13}$$

We try to express the right-hand side above in terms of the gaps $\Delta_{n-1,m}$ and $\Delta_{n-1,m-1}$ for $n-1$ particles. This requires the functions to be orthogonal to 1 in the space $L^2(\mathbb{R}^{n-1}, \Gamma_{n-1}(\hat{v}_k))$, where $\hat{v}_k$ is obtained from v by removing the component v_k. For this purpose, we define the projections

$$\pi_k[h] := \int h\Gamma_{n-1}(\hat{v_k})\, dv_1 \dots dv_{k-1}dv_{k+1}\dots dv_n.$$

and for each k we replace $\langle h, \bar{L}^{(k)}_{n,m}[h]\rangle$ by $\langle (h-\pi_k h), \bar{L}^{(k)}_{n,m}(h-\pi_k h)\rangle$. We can do this since $\bar{L}^{(k)}_{n,m}$ is self-adjoint and the range of π_k is exactly the kernel of $\bar{L}^{(k)}_{n,m}$. Thus, from (2.13),

$$\Delta_{n,m} = \frac{1}{n-1}\inf\sum_{k=1}^{n}\langle (h-\pi_k h), L^{(k)}_{n,m}(h-\pi_k h)\rangle \tag{2.14}$$

where each infimum is over $\{h \in L^2(\mathbb{R}^n, \Gamma_n) : \int_{\mathbb{R}^n} h(v)\Gamma_n = 0 \text{ and } \int_{\mathbb{R}^n} h(v)^2\Gamma_n\, dv = 1\}$. Since $(h-\pi_k h)$ is orthogonal to the constant function 1 in $L^2(\mathbb{R}^{n-1}, \Gamma(\hat{v}_k))$ by construction, the definition of the spectral gap implies

$$\begin{aligned}
\Delta_{n,m} &\geq \frac{1}{n-1}\inf\left(\sum_{k=m+1}^{n}\Delta_{n-1,m}(||h-\pi_k h||^2) + \sum_{k=1}^{m}\Delta_{n-1,m-1}(||h-\pi_k h||^2)\right) \text{ (by Remark 1)}\\
&= \frac{1}{N-1}\inf\left(\Delta_{n-1,m}\sum_{k=m+1}^{n}(||h||^2 - ||\pi_k h||^2) + \Delta_{n-1,m-1}\sum_{k=1}^{m}(||h||^2 - ||\pi_k h||^2)\right)\\
&\geq \frac{n-m}{n-1}\Delta_{n-1,m} + \frac{m}{n-1}\Delta_{n-1,m-1} - \frac{1}{n-1}\max\{\Delta_{n-1,m}, \Delta_{n-1,m-1}\}\sup\sum_{k=1}^{n}||\pi_k h||^2\ ,
\end{aligned}$$

here we have used symmetry among $1, ..., m$ and $m+1, ..., n$ and the fact that the infimum is over functions with norm 1.

First, we note that $\Delta_{n-1,m} \geq \Delta_{n-1,m-1}$ since $(I - \bar{P}_m) \geq 0$. Next, $\sup\Big\{\sum_{k=1}^n ||\pi_k h||^2_{L^2(\Gamma_n)},\ ||h||_{L^2(\Gamma_n)} = 1\Big\}$ equals $\sup\langle h, \sum_{k=1}^n \pi_k h\rangle$. Since $\{\pi_k\}_1^n$ is a collection of commuting projection operators, $\sum_{k=1}^n \pi_k$ is a projection and the supremum is 1.

We then get

$$\Delta_{n,m} \geq \frac{n-m}{n-1}\Delta_{n-1,m} + \frac{m}{n-1}\Delta_{n-1,m-1} - \frac{1}{n-1}\Delta_{n-1,m},$$

which implies claim (2.11).

We now prove the first inequality in Theorem 2. The region of interest is $\{(n,m) : 1 \leq m \leq n-1\}$. We will use induction on $n \geq 2$.

- The base case $n=2$, $m=1$ is the trivial statement $\Delta_{2,1} \geq \Delta_{2,1}$. We know $\Delta_{n,0} \geq 0$ and thus plugging in $m=1$ in (2.11), we see that

$$\Delta_{n,1} \geq \Delta_{2,1} \frac{1}{n-1}. \tag{2.15}$$

- Now suppose

$$\Delta_{n,m} \geq \Delta_{2,1} \frac{m}{n-1} \tag{2.16}$$

for all m such that $1 \leq m \leq n-1$. To show that $\Delta_{n+1,m} \geq \Delta_{2,1} \frac{m}{n}$ for all m such that $1 \leq m \leq n$, we consider the following two cases:

1. $m = 1$: We need to show that $\Delta_{n+1,1} \geq \frac{\Delta_{2,1}}{N}$. From (2.11), we deduce that

$$\Delta_{n+1,1} \geq \frac{n-1}{n} \Delta_{n,1} + \frac{1}{n} \Delta_{n,0} = \frac{n-1}{n} \Delta_{n,1} \ .$$

   Applying (2.16) with $m = 1$ then completes the proof of this case.

2. $1 < m \leq n$:

$$\begin{aligned} \Delta_{n+1,m} &\geq \frac{n-m}{n} \left(\frac{m\Delta_{2,1}}{n-1} \right) + \frac{m}{n} \left(\frac{(m-1)\Delta_{2,1}}{n-1} \right) \text{ (using (2.11) and (2.16))} \\ &= \Delta_{2,1} \frac{m}{n(n-1)} (n - m + m - 1) = \Delta_{2,1} \frac{m}{n}. \end{aligned}$$

This proves the first inequality in (2.10). We prove the second inequality in (2.10), by finding an upper bound proportional to $\frac{m}{n-1}$, for $\Delta_{n,m}$. This can be done by finding a (possibly crude) upper bound on the eigenvalues of $L_{n,m}$ on the space of second degree Hermite polynomials with weight Γ. This space is invariant under $L_{n,m}$ and its action on it with basis $\{\sum_{k=m+1}^{n} H_2(v_k), \sum_{k=1}^{m} H_2(v_k)\}$ can be described by the following matrix (as mentioned before, this is related to the evolution of kinetic energy of the system). We use the identities $Q_{ij}H_2(v_i) = (H_2(v_i) + H_2(v_j))/2$ and $Q_{ij}H_2(v_k) = H_2(v_k)$ for $i, j \neq k$ in obtaining the entries.

$$\begin{pmatrix} \frac{\lambda m}{n-1} & \frac{-\lambda m}{n-1} \\ -\frac{\lambda(n-m)}{n-1} & \frac{\lambda(n-m)}{n-1} + \mu \end{pmatrix}.$$

Its smallest eigenvalue is $\frac{1}{2}(\mu + \frac{n\lambda}{n-1})\big(1 - \sqrt{1 - \frac{4m\lambda\mu}{n-1} \frac{1}{(\mu + \frac{n\lambda}{n-1})^2}}\big)$. Hence, by definition of the gap,

$$\Delta_{n,m} \leq \frac{1}{2}(\mu + \frac{n\lambda}{n-1})\big(1 - \sqrt{1 - \frac{m}{n-1}\frac{4\lambda\mu}{(\mu + \frac{n\lambda}{n-1})^2}}\big)$$

For n large enough, we can write

$$\Delta_{n,m} \leq \frac{1}{2}\left(\mu + \frac{n\lambda}{n-1}\right)\frac{m}{n-1}\frac{4\lambda\mu}{(\mu + \frac{n\lambda}{n-1})^2}$$

or

$$\Delta_{n,m} \leq \frac{m}{n-1}\frac{2\lambda\mu}{\mu + \lambda}$$

□

The spectral gap has the nice $\frac{m}{n}$ scaling, which is a macroscopic quantity. When $m = n$, $\Delta_{n,m}$ has the same order of magnitude as the fully thermostated model in [2], and when $m = 0$, this gap is zero because the eigenvalue 0 is degenerate since any radial function on $\mathbb{R}^n$ is in the kernel of $L_{n,0}$. Our next step is to study the relative entropy for the partially thermostated Kac model.

Approach to Equilibrium in Relative Entropy

Ideally, we would expect a decay of the form $S(e^{-tL_{n,m}}f|\Gamma_n) \leq Ae^{-\frac{m}{n}t}S(f|\Gamma_n)$. We showed a weaker decay rate of the form $S(e^{-tL_{n,m}}f|\Gamma_n) \leq Ae^{-\frac{m}{n^2}t}$. Our main result is summarized in the following theorem:

Theorem 3 *Assume $1 \leq m < n$ and let $h(v,t)$ be the solution of (2.7). Then we have that*

$$S(h(v,t)) \leq \left(-\frac{\delta_- e^{-\delta_+ t}}{\delta_+ - \delta_-} + \frac{\delta_+ e^{-\delta_- t}}{\delta_+ - \delta_-}\right) S(h(v,0)) \tag{2.17}$$

where $\delta_\pm \equiv \delta_\pm(n,m) = \left(\frac{n\lambda+\mu}{2} \pm \frac{1}{2}\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}\right)$.

Note that there is equality in (2.17) when $t = 0$.

Since only the first m particles are thermostated, the Kac rotations are necessary to push the $n-m$ non-thermostated particles to the Gaussian state. One way the contribution of the $Q_{i,j}$-s can be quantified is the relation

$$P_1 Q_{1,2} P_1 \psi = P_1 M_2[\psi] \tag{2.18}$$

which follows from

$$Q_{1,2}P_1\psi(v_1,v_2) = \int_{w\in\mathbb{R}} \fint_{\theta=0}^{2\pi} \psi(w, v_1\cos\theta + v_2\sin\theta)\, g(v_1\cos\theta - v_2\sin\theta)\, dw\, d\theta,$$

and

$$\begin{aligned}
P_1[Q_{1,2}P_1\psi](v_1,v_2) &= g(v_1)\int_{\mathbb{R}}\int_{\mathbb{R}}\fint_0^{2\pi} \psi(w_2, w_1\cos\theta + v_2\sin\theta)\, g(w_1\cos\theta - v_2\sin\theta)\, d\theta\, dw_2\, dw_1 \\
&= g(v_1)\int_{\mathbb{R}}\int_{\mathbb{R}}\fint_0^{2\pi} \psi(w_2, w_1\cos\theta + v_2\sin\theta)\, g(w_1\cos\theta - v_2\sin\theta)\, d\theta\, dw_1\, dw_2 \\
&= g(v_1)\int_{\mathbb{R}}\int_{\mathbb{R}}\fint_0^{2\pi} M_2[\psi](w_2,v_2)\, dw_2 \\
&= P_1[M_2[\psi]] = M_2[P_1[\psi]].
\end{aligned}$$

This is good news because all particles will feel a strong or Maxwellian thermostat after three or more Kac collisions. The proof combines the convexity of the relative entropy with Dyson's expansion with Q as the perturbative term. This Dyson expansion is given by

$$\begin{aligned}
e^{-(a+b)t}e^{(aQ+b\frac{1}{m}\sum_{i=1}^m \bar{P}_i)t}h = e^{-at}\Big\{ &\{e^{\frac{b}{m}\sum(\bar{P}_k-I)t} + \int_0^t dt_1 e^{\frac{b}{m}\sum(\bar{P}_k-I)(t-t_1)}\ aQ\ e^{\frac{b}{m}\sum(\bar{P}_k-I)t_1} \\
&+ \int_0^t dt_1 \int_0^{t_1} dt_2\ e^{\frac{b}{m}\sum(\bar{P}_k-I)(t-t_1)}\ aQ\ e^{\frac{b}{m}\sum(\bar{P}_k-I)(t_1-t_2)}\ aQ\ e^{\frac{b}{m}\sum(\bar{P}_k-I)t_2} + \ldots \Big\}.
\end{aligned}$$

to prove Theorem 3 from upper bounds on $S(e^{tb\sum_k(\bar{P}_k-I)}Q[h])$. We reach this goal in Lemma 4 by building up the upper bound on $S(P_1Q[h])$ of the following lemma.

Lemma 3

$$\sum_{j=2}^n S(P_1Q_{1,j}[h]) \le \left(n-1-\frac{1}{2}\right)S(h) \tag{2.19}$$

Remark 2 *The "$-\frac{1}{2}$" term will provide a contraction.*

Before proving this lemma, we give Han's inequality [11] and its derivation from the Loomis-Whitney inequality adapted from [2] (Lemma 2.12). The Loomis-Whitney inequality says

$$||\prod_{i=1}^n h_i(v^{(i)})||_{L^1(\mu_n)} \le \prod_{j=1}^n ||h_i||_{L^{n-1}(\mu_{n-1})},$$

here $\mu_n = \prod \mu_0(dv_i)$ is a product measure. Han's inequality for our purposes says for that if $h \geq$ belongs to $L^1(\Gamma_n)$ and lies in the Orlicz space $L\log^+ L(\Gamma_n)$ [2], we have

$$\sum_{i=1}^{n} S\left(\bar{P}_i[h]\Gamma_n \middle| \Gamma_n\right) \leq (n-1)S(h\Gamma_n|\Gamma_n). \tag{2.20}$$

Proof (of the Han inequality).- Let $Z = \int_{\mathbb{R}^n} \prod \bar{P}_j[h]^{1/(n-1)}\,\Gamma_{n,\beta}(v)dv$. Then, by the Loomis-Whitney inequality we have

$$Z \leq \prod_{i=1}^{n} \left\{ \int \bar{P}_i[h](v^{(i)})\Gamma_{n-1,\beta}(v^{(i)}) \right\}^{\frac{1}{n-1}} = 1.$$

Using this idea together with Jensen's inequality on $\psi \mapsto \int \psi \ln(\psi)\nu(dv)$ with $\nu(dv) = Z^{-1}\prod_{i=1}^{n}\left\{\int \bar{P}_i[h](v^{(i)})\Gamma_{n-1,\beta}(v^{(i)})\right\}^{\frac{1}{n-1}} dv$ gives us Han's inequality as follows.

$$\begin{aligned}
S(h) &= \int_{\mathbb{R}^n} h\ln[h]\Gamma_{n,\beta}(v)\,dv \\
&= \int_{\mathbb{R}^n} h\ln\left[\frac{[h]}{\prod \bar{P}_j[h]^{1/(n-1)}}\right]\Gamma_{n,\beta}(v)\,dv + \int_{\mathbb{R}^n} h\ln\left[\prod \bar{P}_j[h]^{1/(n-1)}\right]\Gamma_{n,\beta}(v)\,dv \\
&= Z\int_{\mathbb{R}^n}\left[\frac{[h]}{\prod \bar{P}_j[h]^{1/(n-1)}}\right]\ln\left[\frac{[h]}{\prod \bar{P}_j[h]^{1/(n-1)}}\right]\frac{\prod \bar{P}_j[h]^{1/(n-1)}}{Z}\Gamma_{n,\beta}(v)\,dv \\
&\quad + \frac{1}{n-1}\sum_{i=1}^{n}\int_{\mathbb{R}^n} h\ln[\bar{P}_i[h]]\,\Gamma_{n,\beta} \\
&\geq Z\left(\frac{1}{Z}\ln\frac{1}{Z}\right) + \frac{1}{n-1}S(\bar{P}_i h) \geq \frac{1}{n-1}S(\bar{P}_i h).
\end{aligned}$$

If h is symmetric in all its variables, then Han's inequality implies that

$$S(\bar{P}_j[h]) \leq (1-\frac{1}{n})S(h) \text{ for each } j. \tag{2.21}$$

We are now ready for the proof of Lemma 3.

[2] i.e. $\int h\log^+[h]\Gamma < \infty$.

Proof.- For $j > 1$,

$$
\begin{aligned}
S(\bar{P}_1 Q_{1j} h) &= \int \bar{P}_1 Q_{1j} h \log\left(\bar{P}_1 Q_{1j} h\right)\ \Gamma dv \\
&= \int \bar{P}_1 (\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j [h]}) \log\left(\bar{P}_1 (\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h})\right) \bar{P}_1 \bar{P}_j h\ \Gamma dv + \int \bar{P}_1 Q_{1j} h \log(\bar{P}_1 \bar{P}_j h)\ \Gamma dv
\end{aligned}
$$

where we use the fact that $\bar{P}_1 \bar{P}_j h$ does not depend on v_1. Since the argument of the logarithm in the last term is also independent of v_j, we can integrate $\bar{P}_1 Q_{1j} h$ with respect to v_1 and v_j and use the identity $\int P_1 Q_{1j} h\ \Gamma_2(v_1, v_j) dv_1 dv_j = \int h\ \Gamma_2(v_1, v_j) dv_1 dv_j = \bar{P}_1 \bar{P}_j h$ to write:

$$
S(\bar{P}_1 Q_{1j} h) = \int \bar{P}_1 (\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h}) \log\left(\bar{P}_1 (\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h})\right) \bar{P}_1 \bar{P}_j h\ \Gamma dv + \int \bar{P}_1 \bar{P}_j h \log(\bar{P}_1 \bar{P}_j h)\ \Gamma dv.
$$

Now, we apply the symmetric version of Han's inequality (2.21) to $\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h}$ as a function of v_1 and v_j to get:

$$
\begin{aligned}
S(P_1 Q_{1j} h) &\le \frac{1}{2} \int \frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h} \log\left(\frac{Q_{1j} h}{\bar{P}_1 \bar{P}_j h}\right) \bar{P}_1 \bar{P}_j h\ \Gamma dv + \int \bar{P}_1 \bar{P}_j h \log(\bar{P}_1 \bar{P}_j h)\ \Gamma dv \\
&= \frac{1}{2} S(Q_{1j} h) - \frac{1}{2} \int Q_{1j} h \log(\bar{P}_1 \bar{P}_j h)\ \Gamma dv + \int \bar{P}_1 \bar{P}_j h \log(\bar{P}_1 \bar{P}_j h)\ \Gamma dv \\
&= \frac{1}{2} S(Q_{1j} h) + \frac{1}{2} S(\bar{P}_1 \bar{P}_j h)
\end{aligned}
$$

where the self-adjointness of Q_{1j} and the independence of $\bar{P}_1 \bar{P}_j [h]$ on v_1 and v_j was used to arrive to the last step. Finally, summing these terms and noting that $S(Q_{1j} h) \le S(h)$, by the averaging property of Q_{1j}, we get

$$
\sum_{j=2}^{n} S(\bar{P}_1 Q_{1j} h) \le \frac{n-1}{2} S(h) + \frac{1}{2} \sum_{j=2}^{n} S(\bar{P}_j \bar{P}_1 h)\,.
$$

Han's inequality (2.20) to $\bar{P}_1 h \equiv (\bar{P}_1 h)(v_2, ... v_n)$ completes the proof. $\square$

As a corollary of this lemma we obtain the inequality

$$
S(\bar{P}_1 Q[h]) \le (1 - \frac{2}{n} + \binom{n}{2}^{-1} (\frac{n-3}{2})) S(h) = \left(1 - \frac{1}{n(n-1)}\right) S(h), \tag{2.22}
$$

which follows from the convexity property of S and bounding from above by $S(h)$ all the $S(\bar{P}_1 Q_{i,j}[h])$ terms with $1 < i, j$. And now we are ready to give Lemma 4.

Lemma 4 *Let* $1 \leq m < n$. *Then*

$$S\left(\exp\left(\mu \sum_{k=1}^{m} (\bar{P}_k - I)t\right) Qh\right) \leq \left(1 - \frac{m(1-e^{-\mu t})}{n(n-1)}\right) S(h). \tag{2.23}$$

Proof.- We prove the above by induction on m. The base case $m = 1$ (and any $n > 1$) follows from the observation that $\bar{P}_1$ is a projection. Thus we have

$$e^{\mu t(\bar{P}_1 - I)} = e^{-\mu t} I + (1 - e^{-\mu t}) \bar{P}_1, \tag{2.24}$$

and it follows that

$$\begin{aligned}
S(e^{\mu(\bar{P}_1 - I)t} Qh) &= S(e^{-\mu t} Qh + (1-e^{-\mu t}) \bar{P}_1 Qh) \qquad \text{(since } \bar{P}_1 \text{ is a projection)} \\
&\leq e^{-\mu t} S(Qh) + (1-e^{-\mu t}) S(\bar{P}_1 Qh) \\
&\leq e^{-\mu t} S(h) + (1-e^{-\mu t}) \frac{1}{\binom{n}{2}} \sum_{i<j} S(\bar{P}_1 Q_{ij} h) \\
&= e^{-\mu t} S(h) + (1-e^{-\mu t}) \frac{1}{\binom{n}{2}} \Big(\sum_{i<j, i,j \neq 1} S(\bar{P}_1 Q_{ij} h) + \sum_{j=2}^{n} S(\bar{P}_1 Q_{1j} h) \Big) \\
&\leq e^{-\mu t} S(h) + (1-e^{-\mu t}) \frac{1}{\binom{n}{2}} \Big(\sum_{i<j, i,j \neq 1} S(h) + (n - 1 - \frac{1}{2}) S(h) \Big) \\
&= \left(1 - \frac{1-e^{-\mu t}}{n(n-1)}\right) S(h).
\end{aligned}$$

Here we used Han's inequality in the last inequality and used the convexity of the entropy and the averaging property of $\bar{P}_1$ and Q in the previous steps.

For our induction we consider the set $\{(n, m) : 2 \leq m < n\}$. W assume that the lemma is true for $m - 1$ (and any $n > m - 1$). To show that the statement is true for case m, we analyze the entropy of $\bar{P}_m \exp\big(\mu \sum_{k=1}^{m-1} (\bar{P}_k - I)t\big)[h]$. We expand the Kac operator Q and split it into terms that contain the variable m and those that do not. We again use the convexity of the entropy.

$$S\left(\bar{P}_m \exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k - I)t\big)Qh\right) \leq (1-\frac{2}{n})S\left(\frac{\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)}{\binom{n-1}{2}}\sum_{\substack{i<j\\ i,j\neq m}} Q_{ij}\bar{P}_m h\right)$$
$$+\frac{2}{n}S\left(\frac{\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)}{n-1}\bar{P}_m\sum_{l\neq m}Q_{lm}h\right).$$

In the first term[3], we also use the fact that P_m commutes with $Q_{i,j}$ when neither of i or j equals m. Next, we treat the terms as follows:

- First Term: We apply the induction hypothesis for $m-1$, $n-1$ since $\bar{P}_m h$ is a function of $n-1$ variables and $\binom{n-1}{2}^{-1}\sum_{\substack{i<j\\ i,j\neq m}} Q_{ij}$ is the Kac operator acting on $n-1$ variables.
- Second Term: We use the averaging property of $\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)$, convexity of entropy, and equation 2.19.

We obtain

$$S(\bar{P}_m \exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)Qh) \leq (1-\frac{2}{n})\left(1-(m-1)\frac{1-e^{-\mu t}}{(n-1)(n-2)}\right)S(h)+\frac{2}{n}\frac{1}{n-1}(n-\frac{3}{2})S(h)\,. \tag{2.25}$$

Now starting with the left-hand side of (2.23) and using convexity property of S plus the fact that $\bar{P}_m$ is a projection, we write

$$\begin{aligned}
S\left(\exp(\mu\sum_{k=1}^{m}(\bar{P}_k-I)t)Qh\right) &= S\left((e^{-\mu t}I+(1-e^{-\mu t})\bar{P}_m)\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)Qh\right)\\
&\leq e^{-\mu t}S\left(\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)Qh\right)\\
&\quad+(1-e^{-\mu t})S\left(\bar{P}_m\exp\big(\mu\sum_{k=1}^{m-1}(\bar{P}_k-I)t\big)Qh\right)
\end{aligned}$$

The lemma follows after applying the induction hypothesis for the case $m-1, n$ for the first term, and the bound (2.25) for the second term. □

[3] This term is non-zero only when $n>2$, which is the case here.

Proof (of Theorem 3).- Expanding $e^{-L_{n,m}t}$ using the Dyson series with Q as the perturbation:

$$
\begin{aligned}
e^{n\lambda(Q-I)t+\mu\sum_k(\bar{P}_k-I)t} &= e^{-n\lambda t}e^{n\lambda Qt+\mu\sum(\bar{P}_k-I)t} \\
&= e^{-n\lambda t}\{e^{\mu\sum(\bar{P}_k-I)t} + \int_0^t dt_1 e^{\mu\sum(\bar{P}_k-I)(t-t_1)}\ n\lambda Q\ e^{\mu\sum(\bar{P}_k-I)t_1} \\
&+ \int_0^t dt_1 \int_0^{t_1} dt_2\ e^{\mu\sum(\bar{P}_k-I)(t-t_1)}\ n\lambda Q\ e^{\mu\sum(\bar{P}_k-I)(t_1-t_2)}\ n\lambda Q\ e^{\mu\sum(\bar{P}_k-I)t_2} + ...\}
\end{aligned}
$$

Therefore, using the convexity of entropy, and Lemma 4,

$$
\begin{aligned}
S(h(.,t)) &\leq e^{-n\lambda t}\left(1 + n\lambda\int_0^t dt_1 A(t-t_1) + (n\lambda)^2\int_0^t dt_1\int_0^{t_1} dt_2 A(t-t_1)A(t_1-t_2) + ...\right) S(h(.,0)) \\
&= e^{-n\lambda t}\left(1 + n\lambda(A*1) + (n\lambda)^2(A*A*1) + ...\right) S(h(.,0))
\end{aligned}
$$

where $*$ is the Laplace-convolution operation. Thus we have that

$$
S(h(.,t)) \leq e^{-n\lambda t}\ \varphi(t) S(h(.,0)) \tag{2.26}
$$

where φ is defined through the series in the penultimate inequality. The Laplace transform $\tilde{\varphi}(s)$ of $\varphi(t)$ is given by

$$
\tilde{\varphi}(s) = \frac{1}{s}\ \sum_{k=0}^{\infty}(n\lambda\ \tilde{A}(s))^k,
$$

where $\tilde{A}(s) = \frac{1}{s} - \frac{m}{n(n-1)}(\frac{1}{s} - \frac{1}{s+\mu})$ is the Laplace transform of $A(t)$. The inverse Laplace transform gives

$$
-\frac{\delta_- e^{(n\lambda-\delta_+)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}} + \frac{\delta_+ e^{(n\lambda-\delta_-)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}}.
$$

Since $\varphi(t)$ (see eq. (2.26)) is continuous and has exponential order, we can invoke the uniqueness of the Inverse Laplace Transform (See e.g. [8]) to get $\varphi(t)$. Summing the geometric series on a convergent domain, say $s > n\lambda$, we get

$$
\tilde{\varphi}(s) = \frac{s+\mu}{s^2 + (\mu - n\lambda)s - n\mu\lambda(1 - \frac{m}{n(n-1)})}.
$$

The inverse Laplace transform of the above is

$$
-\frac{\delta_- e^{(n\lambda-\delta_+)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}} + \frac{\delta_+ e^{(n\lambda-\delta_-)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}}.
$$

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$$\varphi(t) = -\frac{\delta_- e^{(n\lambda-\delta_+)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}} + \frac{\delta_+ e^{(n\lambda-\delta_-)t}}{\sqrt{(n\lambda+\mu)^2 - 4m\lambda\mu/(n-1)}}.$$

Plugging this into (2.26), we obtain the desired result. □

This result is suboptimal when $n = m$, as was shown even for a weaker thermostat in [2]. To possibly improve this result, one should avoid the convexity argument $S(\alpha\bar{P}_1 + (1-\alpha)Q[h]) \leq \alpha S(\bar{P}_1[h]) + (1-\alpha)S(Q[h])$. Appendix A2 shows that the right-hand side can be larger than $(2-\epsilon)S(h)$ for any ϵ, showing that this convexity inequality is weak.

This chapter ends with two small theorems that support the claim that the results proven for the strong thermostat can also be proven for the Maxwellian Thermostat. The first result is a van Hove (weak coupling- large time) limit [9] that creates a link connecting the strong thermostat and the weak thermostat on a 2-particle system with $(n, m) = (2, 1)$.The second result is a propagation of chaos statement for the partially thermostated Kac model. The proof of the propagation of chaos given below applies without change if we replace the strong thermostats P_i with the Maxwellian thermostats M_i.

2.3 van Hove Limit

Consider a system of 2 particles where the first particle interacts with a stong thermostat. The master equation is given by (1.18). If we increase the rate of action of the thermostat then particle 1 is given a Gaussian distribution $\Gamma_{1,\beta}$ very frequently and after a Kac collision $Q_{1,2}$ with particle 1, particle 2 feels as if it interacted with the weak thermostat at the same temperature $\frac{1}{\beta}$. The rate can be increased by letting the time scale $\frac{1}{2\lambda}$ of the Kac collision be very large and sampling very slowly so that the new time variable is $\tau = 2\lambda t$. The strong thermostat now acts at a very small time scale. We are interested in $\tilde{f}^\lambda$ given by

$$\tilde{f}^\lambda(v_1, v_2, \tau) = f^\lambda(v_1, v_2, \frac{\tau}{\lambda}).$$

It satisfies the equation

$$\frac{\partial \tilde{f}^\lambda}{\partial \tau} = -2(I - Q_{12})\tilde{f}^\lambda - \frac{\mu}{\lambda}(I - P_1)\tilde{f}^\lambda =: -\frac{\mathcal{G}^\lambda}{\lambda}\tilde{f}^\lambda. \tag{2.27}$$

Theorem 4 *Let $\tilde{f}^\lambda$ satisfy the following equation* (2.27) *with initial condition $\tilde{f}^\lambda(v_1, v_2, 0) = \phi(v_1, v_2) \in L^1(\mathbb{R}^2)$. Then, for any $\tau > 0$ fixed, $\overset{L^1}{\lim\limits_{\lambda\to 0}} \tilde{f}^\lambda$ exists and has the form $g(v_1)\tilde{f}(v_2, \tau)$ where $\tilde{f}$ satisfies the equation*

$$\frac{\partial \tilde{f}}{\partial \tau} = -2(I - M_2)\tilde{f} \tag{2.28}$$

together with the initial condition $\tilde{f}(v_2, 0) = \int \phi(v_1, v_2)\, dv_1$. M_2 is the weak thermostat given by equation (1.15).

Proof.- For each λ, $|\frac{\mathcal{G}^\lambda}{\lambda}| \leq (4 + 2\mu/\lambda)$. So we can use Dyson's expansion (an infinite series version of Duhamel's formula). This gives us:

$$e^{-\frac{\tau\mathcal{G}^\lambda}{\lambda}}\phi = \sum_{k=0}^{\infty} a_k(\phi),$$

where

$$\begin{aligned}
a_0(\phi) &= e^{-\frac{\mu}{\lambda}(I-P_1)\tau}\phi, \\
a_1(\phi) &= \int_{t_1=0}^{\tau} e^{-\frac{\mu}{\lambda}(I-P_1)(\tau-t_1)}[-2(I - Q_{1,2})]e^{-\frac{\mu}{\lambda}(I-P_1)t_1}\phi dt_1 \\
a_k(\phi) &= \int_{\{0\leq t_1\leq\cdots\leq t_{k_1}\leq\tau\}} e^{-\frac{\mu}{\lambda}(I-P_1)(\tau-t_1)}[-2(I-Q_{1,2})]e^{-\frac{\mu}{\lambda}(I-P_1)(t_1-t_2)}\ldots[-2(I-Q_{1,2})]e^{-\frac{\mu}{\lambda}(I-P_1)t_{k-1}}\phi d\vec{t}
\end{aligned}$$

We now find the limit of each $a_k(\phi)$ as $\lambda \to 0$. Since $(I - P_1)$ is idempotent, we have

$$e^{-\frac{\mu}{\lambda}\tau(I-P_1)} = e^{-\frac{\mu}{\lambda}\tau}I + (1 - e^{-\frac{\mu}{\lambda}\tau})P_1,$$

and we thus have

$$||e^{-\frac{\mu}{\lambda}\tau(I-P_1)}\psi - P_1\psi|| \leq e^{-\frac{\mu}{\lambda}\tau}||(I - P_1)\psi|| \leq 4||\psi||e^{-\frac{\mu}{\lambda}\tau} \tag{2.29}$$

for any ψ in L^1. Using (2.29) and (2.18), we obtain

$$\begin{aligned}\lim_{\lambda\to 0} a_k(\phi) &= \int_{\{0\le t_1\le\cdots\le t_{k-1}\le\tau\}} P_1[-2(I-Q_{1,2})]\ldots[-2(I-Q_{1,2})]P_1\phi\, d\vec{t} \\ &= \frac{(-2\tau)^k}{k!} P_1(I-Q_{1,2})P_1\ldots(I-Q_{1,2})P_1[\phi] \\ &= \frac{(-2\tau)^k}{k!} P_1(I-Q_{1,2})P_1^2\ldots P_1^2(I-Q_{1,2})P_1[\phi] \\ &= \frac{(-2\tau)^k}{k!}(P_1 - P_1Q_{1,2}P_1)^k P_1[\phi] \\ &= \frac{(-2\tau)^k}{k!}(P_1 - M_2P_1)^k P_1[\phi] = \frac{(-2\tau)^k}{k!}(I-M_2)^k\,(P_1[\phi])\end{aligned}$$

where we use the idempotence of P_1 for the third equality. So $\sum_{k=0}^{\infty}\lim_{\lambda\to 0} a_k(\phi) = e^{-2\tau(I-M_2)}[P_1[\phi]]$.

To show that $\lim\limits_{\lambda\to 0} e^{-\frac{\mathcal{G}\lambda}{\lambda}\tau}\phi = e^{-2\tau(I-M_2)}[P_1[\phi]]$, it suffices to show that the series

$$\sum_{k=0}^{\infty} a_k(\phi)$$

converges in L^1 uniformly in λ. This follows from the inequality

$$||a_k(\phi)|| \le \frac{(4\tau)^k}{k!}||\phi||_{L^1}$$

which holds for all λ. $\square$

This proof can be generalized to give the following van Hove results for the n-particle case. Let's use the expresstion "the van Hove limit of $\{A(\lambda)\}_{\lambda>0}$ as $\lambda\to 0$ is A^* with idempotent operator B" to mean

$$\lim_{\lambda\to 0} e^{-\frac{\tau}{\lambda}A(\lambda)}\phi = e^{-\tau A^*}(B\phi) = Be^{-\tau A^*}[\phi] \tag{2.30}$$

for all $\tau > 0$ and all $\phi\in L^1(\mathbb{R}^n)$. The following facts are given without proof

1. The van Hove limit of $\{\lambda\sum_{j=2}^{n}(I-Q_{1,j})+\mu(I-P_1)\}$ acting on $L^1(\mathbb{R}^n)$ is $\sum_{j=2}^{n}(I-M_i)$ with idempotent operator P_1. This means that it suffices to thermostat a single particle strongly so that all the other particles feel thermostated.

2. The van Hove limit of $\{2\alpha\lambda n(I-Q_n)+\mu(I-P_1)\}$ acting on $L^1(\mathbb{R}^n)$ is

$$\frac{2}{n-1}\sum_{j=2}^{n}(I-M_i) + \frac{n-2}{n-1}(n-1)(I-Q^{(1)})$$

with idempotent operator P_1. Here $Q^{(1)} = \binom{n-1}{2}^{-1} \sum_{2 \le i < j} Q_{i,j}$ is the Kac operator acting on particles $2, 3, \ldots, n$.

3. Let $\alpha = \frac{2n-1}{n-1}$. The van Hove limit of $\{\lambda \alpha 2n(I - Q_{2n}) + \mu \sum_{i=n+1}^{2n}\}$ acting on $L^1(\mathbb{R}^{2n})$ is

$$n(I - Q_{(n)}) + \frac{2n}{n-1} \sum_{i=1}^{n} (I - M_i),$$

with idempotent operator $P_{n+1} P_{n+2} \ldots P_{2n}$. Here $Q_{(n)}$ and $Q_{(2n)}$ are the Kac operators acting on particles $v_1, \ldots, v_n$ and $v_1, \ldots, v_{2n}$ respectively.

The last van Hove result says that the fully thermostated Kac model can be obtained as a van Hove limit from a partially thermostated Kac model where only half the particles are strongly thermostated.

2.4 Propagation of Chaos

Theorem 5 *(Propagation of Chaos for the Partially Thermostated Kac Model)*
Set $A = \{i : i \ge 1,$ and $(i \mod n_0) \in \{1, 2, \ldots, m_0\}\}$ and $B = \mathbb{N} - A$. Let $\{f_k \in L^1(\mathbb{R}^{kn_0})\}_{k \ge 1}$ be a family of probability distributions that are symmetric under the exchange of particles with indices in A and under the exchange of particles with indices in B. Let $L_{m,n}$ be as in equation (1.18)*. If*

$$\lim_{k \to \infty} \int_{\mathbb{R}^{kn_0}} f_k(v_1, \ldots, v_{kn_0}) \phi(v_1, \ldots, v_l)\, dv = \int_{\mathbb{R}^{kn_0}} \prod_{i \in A, i \le l} \bar{f}_0(v_i) \prod_{j \in B, j \le l} \bar{\bar{f}}_0(v_j)\, \phi(v_1, \ldots, v_l) dv,$$

for every ϕ in $L^\infty(\mathbb{R}^l)$, then

$$\lim_{k \to \infty} \int_{\mathbb{R}^{kn_0}} e^{-tL_{km_0, kn_0}}[f_k](v_1, \ldots, v_{kn_0}) \phi(v_1, \ldots, v_l)\, dv = \int_{\mathbb{R}^{kn_0}} \prod_{i \in A, i \le l} \bar{f}(t, v_i) \prod_{j \in B, j \le l} \bar{\bar{f}}(t, v_j)\, \phi(v_1, \ldots, v_l)\, dv,$$

for every ϕ in $L^\infty(\mathbb{R}^l)$ where $(\bar{f}, \bar{\bar{f}})$ satisfy the following system of Boltzmann-Kac equations:

$$\begin{cases} \frac{\partial \bar{f}}{\partial t}(t,v) &= 2\lambda \left[\int_{\mathbb{R}} \int_0^{2\pi} \bar{f}(t,v^*)(\alpha \bar{f}(t,w^*) + (1-\alpha)\bar{\bar{f}}(t,w^*))\, d\theta\, dw - \bar{f}(t,v) \right] + \eta(P_1 - I)\bar{f} \\ \frac{\partial \bar{\bar{f}}}{\partial t}(t,v) &= 2\lambda \left[\int_{\mathbb{R}} \int_0^{2\pi} \bar{\bar{f}}(t,v^*)(\alpha \bar{f}(t,w^*) + (1-\alpha)\bar{\bar{f}}(t,w^*))\, d\theta\, dw - \bar{\bar{f}}(t,v) \right] \end{cases}, \tag{2.31}$$

together with the initial conditions $(\bar{f}(t=0), \bar{\bar{f}}(t=0)) = (\bar{f}_0, \bar{\bar{f}}_0)$.

This roughly says that a given particle collides with a thermostated particle a fraction α of the time, and with non-thermostated particles the fraction $1-\alpha$ of the time.

Proof.- McKean in [20] gave a short algebraic proof of propagation of chaos for Kac's original model on S^{n-1}. This proof was adapted in [2] to give a propagation of chaos result for the fully thermostated Kac model. This section describes how his proof can be further modified to give a propagation of chaos result for the partially thermostated Kac model in [25].

McKean showed that if $Z = Z(\mathbb{R}^\infty, \text{ symm})$ is the space of bounded and continuous functions that depend on a finite but an arbitrary number of variables, endowed with the product

$$f \otimes g(v_1, \ldots, v_a, v_{a+1}, \ldots, v_{a+b}) = \frac{1}{(a+b)!} \sum_{\sigma} f(v_{\sigma(1)}, \ldots, v_{\sigma(a)}) g(v_{\sigma(a+1)}, \ldots, v_{\sigma(a+b)})$$

and where functions with the same symmetrization are identified; that is, we consider $f = g$ if

$$\int_{\mathbb{R}^\infty} f\phi\, dv = \int_{\mathbb{R}^\infty} g\phi\, dv \text{ for all } \phi \in L^1(\mathbb{R}^\infty) \text{ symmetric in its variables;}$$

then $n\lambda(Q-I)$ can be approximated by $2\lambda\Gamma$ that takes functions depending on k variables to functions depending on $k+1$ variables. Γ, defined by

$$\Gamma[\phi](v_1, \ldots, v_k) = \sum_{i \leq k} \int_0^{2\pi} \left[\phi(v_1, \ldots, v_i \cos\theta - v_{k+1} \sin\theta, v_{i+1}, \ldots, v_k)) - \phi \right] d\theta,$$

is a derivation. That is, $\Gamma[f \otimes g] = \Gamma[f] \otimes g + f \otimes \Gamma[g]$. McKean used this fact together with the Taylor expansion of $e^{tn(Q-I)}$ to prove Kac's propagation of chaos result. The same proof was used in [2] to show that propagation of chaos holds for the fully thermostated Kac model. The observation there is that the generator $-L = \eta \sum_{i=1}^n (M_i - I) + n\lambda(Q-I)$ can be approximated by $\eta \sum_{i=1}^\infty (M_i - I) + 2\lambda\Gamma$ which is a derivation.

McKean's proof works both on S^{n-1} and $\mathbb{R}^n$. We will tweak this construction for the partially thermostated Kac model. Suppose $\alpha = \frac{m_0}{n_0}$ is the fraction of thermostated particles. By thermostating only some of the particles, we divide the indices $1, \ldots, n$ into two groups A_n (the thermostated) and B_n (the rest). We take the initial condition $f_n(0, .)$ to be symmetric under the exchange of particles in A_n and under the exchange of particles in B_n. We want to have a space similar to Z and a derivation similar to Γ that adapts to the fact a new particle introduced in a distribution does not always have to be thermostated.

One approach is to let the underlying space be $\bar{Z} = \bar{Z}((\mathbb{R}^{n_0})^\infty)$ and to let f, g all depend on kn_0 , ln_0 variables. We can let every particle with index $i \equiv 1, 2, \ldots, m_0(\,mod\, n_0)$ to be thermostated. $f \otimes g$ can be defined analogously by

$$f \otimes g(v_1, \ldots, v_{kn_0}, v_{kn_0+1}, \ldots, v_{(k+l)n_0}) = \frac{\sum_\sigma f(v_{\sigma(1)}, \ldots, v_{\sigma(kn_0)}) g(v_{\sigma(kn_0+1)}, \ldots, v_{\sigma((k+l)n_0)})}{((k+l)m_0)!((k+l)(n_0 - m_0))!},$$

where the sum is over all permutations σ that leave A_n (and also B_n) invariant. Then our generator becomes

$$\mathcal{L}_k = kn_0\lambda(Q - I) + \eta \sum_{i=1}^{kn_0} \mathbf{1}_{[1,\ldots,m_0]}(i \mod n_0)(P_i - I).$$

We replace Γ by $\bar{\Gamma} : \bar{Z} \mapsto \bar{Z}$, that takes functions depending on kn_0 variables to functions depending on $(k+1)n_0$ variables, given by the formula

$$\bar{\Gamma}[\phi](v_1, \ldots, v_{(k+1)n_0}) = \sum_{i \le kn_0} \sum_{l=kn_0+1}^{(k+1)n_0} \fint_0^{2\pi} \left[\phi(v_1, \ldots, v_i \cos\theta - v_{k+1} \sin\theta, v_{i+1}, \ldots, v_k) - \phi\right] d\theta$$

It follows that $2\lambda\bar{\Gamma} + \eta \sum_{i=1}^{kn_0} \mathbf{1}_{[1,\ldots,m_0]}(i \mod n_0)(P_i - I)$ is a derivation and

$$\left|\left|\mathcal{L}_k\phi - 2\lambda\bar{\Gamma}[\phi] - \eta \sum_{i=1}^{kn_0} \mathbf{1}_{[1,\ldots,m_0]}(i \mod n_0)(P_i - I)[\phi]\right|\right| \le \frac{l^2 n_0}{k} 4 \binom{ln_0}{2} ||\phi|| + 2\lambda \frac{ln_0}{kn_0 + 1} ||\bar{\Gamma}\phi||$$

holds whenever ϕ only depends on ln_0 variables with $l < k$. This goes to 0 as $k \to \infty$.

We also have the bound

$$||\mathcal{L}_{k+l} \circ \mathcal{L}_{k+l-1} \circ \cdots \circ \mathcal{L}_{k+1} \circ \mathcal{L}_k f||_\infty \le (4\lambda + 2\eta)^{(l+1)} k(k+1) \ldots (k+l-1) ||f||_\infty,$$

which makes $\sum_{l=1}^{\infty} \frac{t^l}{l!} ||\mathcal{L}_{k+l-1} \circ \cdots \circ \mathcal{L}_{k+1} \circ \mathcal{L}_k f||_\infty$ convergent for all k when $t < \frac{1}{4\lambda+2\eta}$. McKean's proof can be used step by step from this point on (see also Lemma 19 in [2]). Thus, chaos propagates for time up to $\frac{0.9}{4\lambda+2\eta}$. Repeating this process with the time evolved data shows propagation of chaos for all time. □

Chapter 3

Properties of the GTW metric d_2

In this section we introduce the GTW-metric d_2 and give some of its basic properties including the intensivity property (see proposition [**?**]) for d_2 on chaotic sequences that was proven in [3]. Let μ and ν be Borel probability measures on $\mathbb{R}^n$ [1]. The GTW metrics d_α are given by

$$d_\alpha(\mu, \nu) = \sup_{\xi \neq \vec{0}} \frac{|\hat{\mu}(\xi) - \hat{\nu}(\xi)|}{|\xi|^\alpha}. \tag{3.1}$$

Here we use the convention that the Fourier transform of ϕ is $\hat{\phi}(\xi) = \int \phi(v) e^{-2\pi i \xi . v}\, dv$. We will use only d_2, even though analogs of Theorems 8 and 9 are valid for any d_α with $\alpha > 0$.

The GTW metrics $\{d_\alpha\}_{\alpha>0}$ were introduced in [22] in the context of the space homogeneous Kac-Boltzmann equation (1.9) where they helped in showing exponentially fast convergence to equilibrium for the initial data with finite $2 + \epsilon$ moment for some $\epsilon > 0$. d_1 and d_2 were used in [7] to show exponential convergence to steady states for the Kac Boltzmann system coupled to multiple Maxwellian thermostats at different temperatures. Similarly, d_1 and d_2 were used by J. Evans in [23] to show existence and ergodicity of non-equilibrium steady states in the Kac model coupled to multiple thermostats.

If μ and ν are Borel measures with finite second moments and the same mean then $d_2(\mu, \nu)$ is guaranteed to be finite. We now show this. We have

[1] This guarantees that μ and ν have continuous Fourier transforms.

$$
\begin{aligned}
d_2(\mu,\nu) &= \sup_{\vec{\xi}\neq 0} |\xi|^{-2}\left|\int e^{2\pi i v.\xi}(\mu(dv)-\nu(dv))\right| \\
&= \sup_{\vec{\xi}\neq 0} |\xi|^{-2}\left|\int (e^{2\pi i v.\xi}-1-2\pi i v.\xi)(\mu(dv)-\nu(dv))\right| \\
&\leq \sup_{\vec{\xi}\neq 0} \left|\int \frac{(e^{2\pi i v.\xi}-1-2\pi i v.\xi)}{(v.\xi)^2} v^2(\mu(dv)-\nu(dv))\right| \\
&\leq \frac{(2\pi)^2}{2}\int v^2\left|\mu(dv)-\nu(dv)\right|.
\end{aligned}
$$

Here we used $|v|^{-2}|\xi|^{-2} \leq (v.\xi)^{-2}$ and the fact that $\int v_i\mu(dv) = \int v_i\nu(dv)$ for each i.

An important property of the d_2 metric is its convexity. Let $c_i \geq 0$ and $\sum_{i=1}^k c_i = 1$. Let $\{\phi_i, \psi_j : 1 \leq i,j \leq k\}$ be probability distributions with mean zero and with finite second moments. Then

$$
d_2(\sum_{i=1}^k c_i\phi_i, \sum_{j=1}^k c_j\psi_j) \leq \sum_{i=1}^k c_i d_2(\phi_i,\psi_i) \tag{3.2}
$$

This follows from the convexity of $|.|$. Equation (3.2) is also true if $k=\infty$. Frequently we will choose the c_i to be $e^{-N\lambda t}\frac{(N\lambda t)^i}{i!}$. We also have for any i,j,

$$
d_2(Q_{i,j}\phi, Q_{i,j}\psi) \leq d_2(\phi,\psi) \tag{3.3}
$$

and taking the average over all $i<j$ gives $d_2(Q\phi, Q\psi) \leq d_2(\phi,\psi)$. This, together with (3.2) with $\phi_i = Q^i\phi$ and $\psi_i = Q^i\psi$ gives:

$$
d_2(e^{-tL}\phi, e^{-tL}\psi) \leq d_2(\phi,\psi). \tag{3.4}
$$

An interesting feature of the d_2 metric is its intensivity property on chaotic distributions given in [3]:

Proposition 6 *Let $f_1,\ldots,f_n$ and $g_1,\ldots,g_n$ be probability densities on $\mathbb{R}$ with finite second moments and 0 first moment. Then*

$$
d_2(\prod_{i=1}^n f_i(v_i), \prod_{j=1}^n g_j(v_j)) = \max_{i\leq n} d_2(f_i,g_i). \tag{3.5}
$$

Proof.- Using the definition: we have

$$d_2(\prod_{i=1}^{n} f_i(v_i), \prod_{j=1}^{n} g_j(v_j)) = \sup_{|\xi|\neq 0} \frac{|\prod_i \hat{f}_i(\xi_i) - \prod_j \hat{g}_j(\xi_j)|}{\sum \xi_i^2}. \tag{3.6}$$

Taking the supremum over a smaller set $\vec{\xi}$ to be $(0,0,\ldots,0,r,0,\ldots,0)^T$ and noting that $\hat{f}_i(0) = \hat{g}_j(0) = 1$ for all i,j implies $d_2(\prod_{i=1}^{n} f_i(v_i), \prod_{j=1}^{n} g_j(v_j)) \geq d_2(f_i, g_i)$ for each i.

Also, adding and subtracting terms of the form $\prod_{i=1}^{l} \hat{f}_i(\xi_i) \prod_{j=l+1}^{m} \hat{g}_j(\xi_j)$ and using $\hat{f}_i(0) = \hat{g}_j(0) = 1$, we see that the right-hand side of (3.6) is dominated by

$$\sup_{\vec{\xi}\neq 0} \frac{\sum_{i=1}^{n} |\hat{f}(\xi_i) - \hat{g}(\xi_i)|}{\sum_i \xi_i^2} \leq \sup_{\vec{\xi}\neq 0} \sum_{i=1}^{n} a_i \frac{|\hat{f}_i(\xi_i) - \hat{g}_i(\xi_i)|}{\xi_i^2}.$$

where $a_i = \xi_i^2/|\xi|^2$. The a_i-s are nonnegative and sum to 1. Thus, the last supremum is $\max_i \sup_{\xi_i\neq 0} \frac{|\hat{f}_i(\xi_i) - \hat{g}_i(\xi_i)|}{\xi_i^2}$ as desired. $\square$

The next proposition relates the d_2 metric to the Kac evolution. It further elaborates on the intensivity property of the d_2 metric. It is intensive for all measures after a wait time of $O(\ln(n))$.

Proposition 7 *(d_2-energy comparison) Let μ and ν be Borel probability measures on $\mathbb{R}^n$ with $n \geq 2$. Let $\int v\mu(dv) = \vec{0} = \int v\nu(dv)$ and $\int |v|^2(\mu(dv) + \nu(dv)) < \infty$, and let $-L = n(I - Q)$ be the generator of the Kac evolution ($\lambda = 1$). Then*

$$\begin{aligned} d_2(e^{-tL}\mu, R_\mu) \leq \frac{(2\pi)^2}{2} \Bigg[&\left(2 - e^{-\frac{n}{n-1}t}\right) \int \frac{|v|^2}{n} \mu(dv) + \\ &e^{-\frac{n}{n-1}t} \max_i \int v_i^2 \mu(dv) + (n-1)e^{-\frac{4n-6}{n-1}t} \max_{i\neq j} \left| \int v_i\, v_j\, \mu(dv) \right| \Bigg] \end{aligned} \tag{3.7}$$

$$d_2(e^{-tL}\mu, e^{-tL}\nu) \leq \frac{(2\pi)^2}{2} ((n-1)e^{-t} + 1) \int_{\mathbb{R}^n} |\mu(dv) - \nu(dv)| \frac{|v|^2}{n} \tag{3.8}$$

Remark 3 *If μ has mean $\vec{m} \neq \vec{0}$. Then $d_2(\mu, R_\mu) = \infty$ because the angular average R_μ has mean $\vec{0}$. One way around this is to use a centered GTW distance d_2' as in [7] and [23], to handle the $\frac{1}{|\xi|}$ divergence as $\xi \to \vec{0}$ in the expression for d_2. This case is omitted.*

The proof of Proposition 7 relies on the action of the Kac evolution on quadratic polynomials. The following lemma says that after time of order $\ln(n)$, $(v.\xi)^2$ is effectively $\frac{|v|^2}{n}|\xi|^2$.

Lemma 5 *(Kac Action on Quadratic Polynomials) Let $n \geq 2$ and let $-L$ be the generator of the master equation* (1.7) *with $\lambda = 1$. For any $v, \xi \in \mathbb{R}^n$, we have*

$$e^{-tL}(v.\xi)^2 = \left(1 - e^{-\frac{n}{n-1}t}\right)\frac{|v|^2|\xi|^2}{n} + e^{-\frac{n}{n-1}t}\sum_{i=1}^n \xi_i^2 v_i^2 + e^{-\frac{4n-6}{n-1}t}\sum_{i\neq j}\xi_i\xi_j v_i v_j$$

It follows that for all $n \geq 2$ and $t \geq 0$ we have

$$\left|e^{-tL}(v.\xi)^2 - \frac{|v|^2|\xi|^2}{n}\right| \leq e^{-t}\left(1 - \frac{1}{n}\right)|v|^2|\xi|^2.$$

Proof (of Lemma 5).- We look at the action of Q on v_1v_2 and on v_1^2 separately. First,

$$Q_{i,j}v_1v_2 = \begin{cases} 0, & \{i,j\} \cap \{1,2\} \neq \phi \\ v_1v_2, & \text{otherwise} \end{cases}$$

It follows that $e^{-tL}v_1v_2 = e^{-n(1-\frac{\binom{n-2}{2}}{\binom{n}{2}})t}v_1v_2$. Similarly, $Qv_1^2 = (1-\frac{1}{n-1})v_1^2 + \frac{1}{n-1}\frac{|v|^2}{n}$. Thus

$$n(Q-I)\begin{pmatrix} v_1^2 \\ \frac{|v|^2}{n}\end{pmatrix} = \begin{pmatrix} -\frac{n}{n-1} & \frac{n}{n-1} \\ 0 & 0\end{pmatrix}\begin{pmatrix} v_1^2 \\ \frac{|v|^2}{n}\end{pmatrix},$$

and since $\exp\left(t\begin{pmatrix} -\frac{n}{n-1} & \frac{n}{n-1} \\ 0 & 0\end{pmatrix}\right) = \begin{pmatrix} e^{-t\frac{n}{n-1}} & 1 - e^{-t\frac{n}{n-1}} \\ 0 & 1\end{pmatrix}$, we obtain:

$$e^{-tL}v_1^2 = e^{-\frac{n}{n-1}t}v_1^2 + (1 - e^{-\frac{n}{n-1}t})\frac{|v|^2}{n}.$$

From these two identities it follows that

$$e^{-tL}(v.\xi)^2 - \frac{|v|^2|\xi|^2}{n} = e^{-n(1-\frac{\binom{n-2}{2}}{\binom{n}{2}})t}\sum_{i\neq j}\xi_i\xi_j v_i v_j + e^{-\frac{n}{n-1}t}\left(\sum_{i=1}^n \xi_i^2 v_i^2 - \frac{|\xi|^2|v|^2}{n}\right).$$

Next we conclude the second bound. Let $a = n\left(1 - \frac{\binom{n-2}{2}}{\binom{n}{2}}\right) = \frac{4n-6}{n-1}$ and let $b = \frac{n}{n-1}$. We have $a \geq 2b$ when $n \geq 2$. The right-hand side can be written as

$$e^{-at}(v.\xi)^2 + (e^{-bt} - e^{-at})\sum_{i=1}^n \xi_i^2 v_i^2 - \frac{|\xi|^2|v|^2}{n}e^{-bt}.$$

$\square$

We are now ready to prove Proposition 7.

Proof (of energy comparison).- We start with the definition of $d_2(e^{-tL}\mu, R_\mu)$.

$$\begin{aligned} d_2(e^{-tL}\mu, R_\mu) &= \sup_{\xi\neq 0} \frac{1}{|\xi|^2}\left|\int_{\mathbb{R}^n} \left(e^{-tL}\mu(dv) - R_\mu\right) e^{-2\pi i v.\xi}\right| \\ &= \sup_{\xi\neq 0} \frac{1}{|\xi|^2}\left|\int_{\mathbb{R}^n} e^{-tL}\mu(dv) \left(e^{-2\pi i v.\xi} - R_{e^{-2\pi i v.\xi}}(v)\right)\right|. \end{aligned}$$

Here we used the self-adjointness of the radial averaging and the fact that $e^{-tL}\mu$ and μ have the same angular average, and $R_{e^{-2\pi i v.\xi}}$ is the angular average of $\exp(-2\pi i v.\xi)$ on which we study next. For brevity, let R denote $R_{e^{-2\pi i v.\xi}}$. Then R is also the radial averaging of $\cos(2\pi v.\xi)$ and we have

$$\begin{aligned} R(v) &= \fint_{|y|=|v|} \cos(2\pi y_n|\xi|)\, dy = \left|S^{n-1}\right|^{-1} \int_{S^{n-1}} \cos(2\pi |v||\xi| \cos\theta_1)\, d\sigma^n \\ &= \frac{|S^{n-2}|}{|S^{n-1}|} \int_{\theta_1=0}^{\pi} \cos(2\pi|v|\,|\xi| \cos\theta_1) \sin(\theta_1)^{n-2}\, d\theta_1 \\ &= \frac{\int_{\theta_1=0}^{\pi} \cos(2\pi|v|\,|\xi|\cos\theta_1)\sin(\theta_1)^{n-2}\, d\theta_1}{\int_{\theta_1=0}^{\pi} \sin(\theta_1)^{n-2}\, d\theta_1}. \end{aligned}$$

We won't use this fact, but this ratio of integrals is evaluated by Mathematica in terms of the hypergeometric function ${}_0F_1$. ${}_0F_1(a,x) = 1 + \sum_{k=1}^{\infty} \frac{1}{a(1+a)(2+a)...(k-1+a)} \frac{x^k}{k!}$ and $R(v) = {}_0F_1(\frac{n}{2}, -\frac{(2\pi|v||\xi|)^2}{4})$. Going back to $d_2(e^{-tL}\mu, R_\mu)$, we can use the fact that $\int v_i \mu(dv) = 0$ for every variable i, so we can set

$$\begin{aligned} d_2(e^{-tL}\mu, R_\mu) &= \sup_{\xi\neq 0} \frac{1}{|\xi|^2}\left|\int_{\mathbb{R}^n} e^{-tL}\mu(dv) \left(e^{-2\pi i v.\xi} - 1 + 2\pi i v.\xi + 1 - R_{e^{-2\pi i v.\xi}}(v)\right)\right| \\ &\leq \sup_{\xi\neq 0} \int_{\mathbb{R}^n} e^{-tL}\mu(dv) \frac{\left|e^{-2\pi i v.\xi} - 1 + 2\pi i v.\xi\right|}{|\xi|^2} + \int_{\mathbb{R}^n} e^{-tL}\mu(dv) \frac{1-R(v)}{|\xi|^2}\, dv. \end{aligned}$$

Taylor's theorem gives $|e^{ix} - 1 - ix| \leq \frac{1}{2}x^2$ for all $x \in \mathbb{R}$. Thus

$$\sup_{\xi\neq 0} \int_{\mathbb{R}^n} e^{-tL}\mu(dv) \frac{\left|e^{-2\pi i v.\xi} - 1 + 2\pi i v.\xi\right|}{|\xi|^2} \leq \frac{(2\pi)^2}{2} \int_{\mathbb{R}^n} e^{-tL}[\mu(dv)] \frac{(v.\xi)^2}{|\xi|^2}\, dv \tag{3.9}$$

This is of order 1 after time of $O(\ln(n))$ by Lemma 5. We next study the second term $(1 - R(v))/|\xi|^2$. It equals

$$\begin{aligned}
\frac{1}{|\xi|^2}\frac{\int[1-\cos(2\pi|v||\xi|\cos\theta_1)](\sin\theta_1)^{n-2}\,d\theta_1}{\int_0^\pi(\sin\theta_1)^{n-2}\,d\theta_1} &= \frac{1}{|\vec{\xi}|^2}\frac{\int 2\sin^2(\pi|v||\xi|\cos\theta_1)(\sin\theta_1)^{n-2}\,d\theta_1}{\int_0^\pi(\sin\theta_1)^{n-2}\,d\theta_1} \\
&\leq 2\frac{\pi^2|v|^2|\xi|^2}{|\vec{\xi}|^2}\frac{\int_0^\pi\cos\theta_1^2\sin\theta_1^{n-2}\,d\theta_1}{\int_0^\pi\sin\theta_1^{n-2}\,d\theta_1} \\
&= \frac{(2\pi)^2|v|^2}{2n}.
\end{aligned}$$

This, together with Lemma 5, proves (3.7).

To prove (3.8), we need a way to “liberate” e^{-tL} so that Lemma 5 can be used. For $d_2(e^{-tL}\mu, e^{-tL}\nu)$, we have:

$$\begin{aligned}
|\xi|^{-2}\left|e^{-tL}\hat{\mu}(\xi) - e^{-tL}\hat{\nu}(\xi)\right| &= |\xi|^{-2}\left|\int_{\mathbb{R}^n} e^{2\pi iv.\xi}\left(e^{-tL}\mu(dv) - e^{-tL}\nu(dv)\right)\right| \\
&= |\xi|^{-2}\left|\int_{\mathbb{R}^n}[(e^{2\pi iv.\xi} - 1 + 2\pi iv.\xi)]\left(e^{-tL}\mu(dv) - e^{-tL}\nu(dv)\right)\right| \\
&\leq \frac{(2\pi)^2}{2}|\xi|^{-2}\int_{\mathbb{R}^n}(v.\xi)^2\left|e^{-tL}[\mu(dv) - \nu(dv)]\right|
\end{aligned}$$

as in (3.9). We now look at the term $\left|e^{-tL}[\mu(dv) - \nu(dv)]\right|$. We have

$$\begin{aligned}
|[Q_{i,j}\mu](A) - [Q_{i,j}\nu](A)| &= \left|\fint_0^{2\pi}([Q_{i,j}(\theta)\mu](A) - [Q_{i,j}(\theta)\nu](A))\,d\theta\right| \\
&= \left|\fint_0^{2\pi}(\mu[Q_{i,j}(-\theta)(A)] - \nu[Q_{i,j}(-\theta)(A)])\,d\theta\right| \\
&\leq \fint_0^{2\pi}|\mu-\nu|\,(Q_{i,j}(-\theta)[A])\,d\theta \\
&= Q_{i,j}|\mu-\nu|[A]
\end{aligned}$$

for any measurable set A. Here

$$Q_{i,j}(\theta)[A] = \{(v_1, \ldots, v_N) : (v_1, \ldots, v_i \cos\theta - v_j \sin\theta, \ldots, v_i \sin\theta + v_j \cos\theta, \ldots, v_N) \in A\}. \tag{3.10}$$

From the convexity of $|.|$ it follows that $|e^{-tL}[\mu(dv) - \nu(dv)]| \leq e^{-tL}|\mu(dv) - \nu(dv)|$. Thus, we can use the self-adjointness of L to let e^{-tL} act on $(v.\xi)^2$. This allows us to use Lemma 5 and obtain the desired upper bounds related to the second moment:

$$\begin{aligned} |\vec{\xi}|^{-2} \left| e^{-tL}\hat{\mu}(\xi) - e^{-tL}\hat{\nu}(\xi) \right| &\leq \frac{(2\pi)^2}{2} |\vec{\xi}|^{-2} \int_{\mathbb{R}^n} e^{-tL}(v.\xi)^2 \, |\mu(dv) - \nu(dv)| \\ &\leq \frac{(2\pi)^2}{2} ((n-1)e^{-t} + 1) \int_{\mathbb{R}^n} \frac{|v|^2}{n} \, |\mu(dv) - \nu(dv)| \, \square \end{aligned}$$

At $t = 0$, $d_2(\mu, R_\mu)$ can in fact be of order n, because if μ is a measure which is even, symmetric in its variables, has finite second moment and mean $\vec{0}$, with $\int_{\mathbb{R}^n} v_1 v_2 \mu(dv) \neq 0$, then

$$\begin{aligned} d_2(\mu, R_\mu) &\geq \lim_{s \to 0, \vec{\xi} = s(1,1,\ldots,1)} |\xi|^{-2} \left| \int \cos(2\pi v.\xi)[\mu(dv) - R_\mu(dv)] \right| \\ &= \lim_{s \to 0, \vec{\xi} = s(1,1,\ldots,1)} |\xi|^{-2} \left| \int (\cos(2\pi v.\xi) - 1)[\mu(dv) - R_\mu(dv)] \right| \\ &= \frac{(2\pi)^2}{2} \lim_{s \to 0, \vec{\xi} = s(1,1,\ldots,1)} |\xi|^{-2} \left| \int (v.\xi)^2 [\mu(dv) - R_\mu(dv)] \right| \\ &= \frac{(2\pi)^2}{2} \lim_{s \to 0, \vec{\xi} = s(1,1,\ldots,1)} |\xi|^{-2} \left| \int \left[(v.\xi)^2 - \frac{|v|^2 |\xi|^2}{n} \right] \mu(dv) \right| \\ &= \frac{(2\pi)^2}{2} \left| \int v_1 v_2 \, \mu(dv) \right| \lim |\xi|^{-2} |\sum_{i \neq j} \xi_i \xi_j| = \frac{(n-1)(2\pi^2)}{2} \left| \int v_1 v_2 \mu(dv) \right|, \end{aligned}$$

and if μ is a measure concentrated on the line $v_1 = v_2 = \ldots = v_n$, then $\int v_1 v_2 \mu(dv)$ equals $\int v_1^2 \mu(dv)$ and $d_2(\mu, R_\mu)$ becomes a multiple of the total energy. The proposition says that this condition won't last for time longer than $O(\ln(n))$. Also, if μ has mean zero and has all correlations zero (as in (3.6)) then (3.7) shows that $d_2(e^{-tL}\mu, R_\mu)$ is never of order n.

Chapter 4

Finite Reservoir Approximation to the Maxwellian Thermostat

In this chapter we consider a more realistic version of the idealized thermostat used in the fully thermostated Kac model. Suppose we have a reservoir (finite thermostat) composed of $\mathcal{N} >> n$ particles that are undergoing Kac collisions, and suppose that this reservoir is at equilibrium at temperature β^{-1}. That is, initially the reservoir has state $\prod_{j=1}^{\mathcal{N}} \Gamma_{1,\beta}(w_j)$. Here w_j denotes the velocity of the j^{th} particle in the reservoir. The system of n particles initially has a generic state $l_0(v)$ that is independent of the state of the thermostat. The initial state of the FR-system is $l_0(v)\Gamma_{\mathcal{N},\beta}(w)$. Here FR-system stands for the system plus the finite reservoir.

Prior to the coupling of the system with the reservoir, each of the system and the reservoir undergoes Kac collisions on its own. The generator of this process (that does not couple the system with the reservoir) is

$$-\mathcal{L}_K = \frac{2\lambda_S}{n-1}\sum_{i<j\leq n}(Q^S_{i,j} - I) + \frac{2\lambda_R}{\mathcal{N}-1}\sum_{i<j\leq \mathcal{N}}(Q^R_{i,j} - I) =: n\lambda_S(Q_S - I) + \mathcal{N}\lambda_R(Q_R - I).$$

Here, we use the convention that the subscript and superscript S and R stand for "system" and "reservoir" respectively. We will also introduce I and T to stand for interaction terms and thermostat terms. So if $\phi(v, w)$ is any function, then

$$Q^S_{1,2}\phi = \int_0^{2\pi} \phi(v_1^*(\theta), v_2^*(\theta), v_3, \ldots, w_1, \ldots, w_{\mathcal{N}})\, d\theta \tag{4.1}$$

$$Q^R_{1,2}\phi = \int_0^{2\pi} \phi(v_1, \ldots, v_n, w_1^*(\theta), w_2^*(\theta), w_3, \ldots, w_{\mathcal{N}})\, d\theta \tag{4.2}$$

$$Q^I_{2,1}\phi = \int_0^{2\pi} \phi(v_1, v_2^*(\theta), v_3, \ldots, v_n, w_1^*(\theta), w_2, \ldots, w_{\mathcal{N}})\, d\theta. \tag{4.3}$$

Next we introduce an interaction term of the form $-\mathcal{L}_I = c\sum_{i=1}^n \sum_{j=1}^{\mathcal{N}} (Q^I_{i,j} - I)$. The dependence of c on n and $\mathcal{N}$ should be decided based on comparison with the thermostat term

$$-\mathcal{L}_T = \mu \sum_{j=1}^{n} (M_i - I)$$

in the fully thermostated Kac model. $\mathcal{L}_T$ makes each of the n particles in the system interact with a thermostat with inverse temperature β at a rate μ. Similarly, the Kac term $\mathcal{L}_K$ makes particle i in the system collide at a rate $\lambda_S n(n-1)/\binom{n}{2}$, which is $O(1)$. So it is expedient to let the interaction term with generator $\mathcal{L}_I$ make each particle in the system collide at a rate of order 1 (in n) with a reservoir particle. This is achieved by setting $c = \frac{\mu}{\mathcal{N}}$ and thus

$$-\mathcal{L}_I = \frac{\mu}{\mathcal{N}} \sum_{i=1}^{n} \sum_{j=1}^{\mathcal{N}} (Q^I_{i,j} - I) =: \frac{\mu n}{\mathcal{N}} (Q_I - I). \tag{4.4}$$

The operators Q_S, Q_R, and Q_I [1] are all averaging operators. In this chapter we will compare the evolutions of the system with finite reservoir (FR-system) to the evolution of the system that interacts with an ideal (infinite) thermostat (the T-system). The generator for the evolution of the FR-system is

$$-L_{FR} = -\mathcal{L}_K - \mathcal{L}_I, \tag{4.5}$$

while the generator for the thermostated system is

$$-L_T = -\mathcal{L}_K - \mathcal{L}_T. \tag{4.6}$$

L_T acts on functions of n variables. In order that both evolutions become comparable, we will let L_T act on functions of $n + \mathcal{N}$ variables by leaving the $\mathcal{N}$ w-variables intact. So $e^{-tL_T}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)] = [e^{-tL_T} l_0](v)\Gamma_{\mathcal{N},\beta}(w)$. We will compare $e^{-tL_T}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$ with

[1] Q^I has a factor of μ in the paper [3].

$e^{-tL_{FR}}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$. These evolutions have different limits: $e^{-tL_T}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)] \to \Gamma_{n+\mathcal{N},\beta}$ while $e^{-tL_{FR}}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)] \to R$, the angular average of $[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$.

These limits are very close if $\mathcal{N} >> n$. In fact, Appendices A1 and A2 show that if $l_0 = h(v)\Gamma_n(v)$ and $R = h_\infty(v,w)\Gamma_{n+\mathcal{N}}$, then

$$||h_\infty - 1||_{L^2(\Gamma_{n+\mathcal{N}})} \le \sqrt{\frac{n}{\mathcal{N}-2}}||h-1||_{L^2(\Gamma_{n+\mathcal{N}})},$$

and

$$d_2(R, \Gamma_{n+\mathcal{N},\beta}) \le \frac{n}{n+\mathcal{N}} d_2(f_0, \Gamma_{n,\beta}).$$

We show that $e^{-tL_T}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$ and $e^{-tL_{FR}}[l_0(v)\Gamma_{\mathcal{N},\beta}(w)]$ stay close in two distances and that their difference goes to zero as $\mathcal{N} \to \infty$ uniformly in time. The uniformity in time is the significance of the following theorems, since the fact that the n marginal of $e^{-tL_{FR}} l_0\Gamma_{\mathcal{N}}$ converges to the $n-$ marginal of $e^{-tL_T} l_0\Gamma_{\mathcal{N}}$ for fixed t as $\mathcal{N} \to \infty$ is more or less intuitive. Our results are Theorem 6 using the L^2 metric and Theorem 7 using the GTW metric d_2. For convenience, we will set $\beta = 2\pi$ in Theorem 7 to make $\Gamma_{k,\beta}$ invariant under the Fourier transform.

A disturbing aspect of the L^2 metric is that it rules out some physically acceptable states. For example, the Gaussian $\Gamma_{n,\alpha}(v)$ at temperature α^{-1} greater than or equal to twice the temperature of the reservoir ($2\beta^{-1}$) satisfies

$$\Gamma_{n,\alpha}(v) = c\exp[(-\alpha^2+\beta^2)v^2/2]\Gamma_{n,\beta}(v). \quad \text{But } \int_{\mathbb{R}^n} \exp[(-\alpha^2+\beta^2)v^2]\Gamma_{n,\beta}(v)\,dv = \infty.$$

Theorem 6 *Let f_0 be the joint initial distribution of the system and finite reservoir and assume that it has the form*

$$f_0(v,w) = h_0(v)\Gamma_{n+\mathcal{N}}(v,w) \tag{4.7}$$

with $h_0 \in L^2(\mathbb{R}^n, \Gamma(v))$. Then for every $t > 0$ we have

$$\|e^{-L_{FR}t}h_0 - e^{-L_T t}h_0\|_{L^2(\Gamma_{n+\mathcal{N}})} \le \frac{n}{\sqrt{\mathcal{N}}}(1-e^{-\frac{\mu}{2}t})\|h_0 - 1\|_{L^2(\Gamma_n)}\,. \tag{4.8}$$

Theorem 7 *Let $\beta = 2\pi$ and let $f_0(v,w)$, the joint initial distribution of the system and finite reservoir, have the form*

$$f_0(v,w) = l_0(v)\Gamma_{\mathcal{N}}(w),$$

with l_0 symmetric and satisfying the following conditions.

$$\int v_i f(v,w)dvdw = \int w_j f(v,w)dvdw = 0 \qquad \int v_i^2 f(v,w)dvdw < \infty. \tag{4.9}$$

Assume moreover that

$$\int v_i^4 l_0(v)dv = E_4 < \infty, \tag{4.10}$$

then for every $t > 0$ we have

$$d_2\left(e^{-L_{FR}t}f_0, e^{-L_T t}f_0\right) \leq \frac{2n}{\mathcal{N}}\left(1 - e^{-\frac{\mu}{4}t}\right)\sqrt{d_2(l_0,\Gamma_n)(F_4 + d_2(l_0,\Gamma_n))}\,, \tag{4.11}$$

with F_4 at most a linear funtion in E_4: $F_4 = 4(2\pi)^4 \max\{AE_4 + B, 2\}$ where $A = \frac{3\sqrt{5}}{8}$ and $B = \frac{11}{8} + \frac{12\pi + 9\sqrt{2}}{32\pi^2}$.

The factor $(2\pi)^4$ is the result of using the convention $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(v)e^{-2\pi i v.\xi}\,dv$. The bounded fourth moment assumption is necessary for our proof.

When we work in the space $L^2(\Gamma_{n+\mathcal{N}})$, the operator Q_M has to be replaced by $Q_{\bar{M}}$ defined by

$$Q_M[h(v,w)\Gamma(v,w)] = Q_{\bar{M}}[h]\Gamma(v,w), \tag{4.12}$$

as in (2.3). Similarly, L_T by $\bar{L}_T$. We give 2 contracting properties of our averaging operators in the space $L^2(\Gamma_{n+\mathcal{N}})$, and then give their analogs under d_2.

Proposition 8

$$||Q^{\alpha}_{i,j}||_2 = 1 \tag{4.13}$$

for $\alpha = S, R,$ or I.

This is because $Q^{\alpha}_{i,j}$ is a projection onto the set of functions that are independent of rotations of v_i and v_j.

Proposition 9 *If u is an $L^2(\Gamma_n)$ function on $\mathbb{R}^n$ with $\int u(v)\Gamma(v)\,dv = 0$, then $||Q_{\bar{M}}u||_2 \leq \left(1 - \frac{1}{2n}\right)||u||_2$.*

Proof.- Let $u(v) = \sum u^{i_1,\ldots,i_n} H_{i_1}(v_1)H_{i_2}(v_2)\ldots H_{i_n}(v_n)$, be the expansion of u in terms of the Hermite polynomials. We have

$$Q_{\bar{M}}u = \frac{1}{n}\sum_{j=1}^{n} \bar{M}_j u = \frac{1}{n}\sum u^{i_1,\ldots,i_n}(b_{i_1} + \cdots + b_{i_n})H_{i_1}(v_1)H_{i_2}(v_2)\ldots H_{i_n}(v_n).$$

We know from (2.5) that $b_0 = 1$ and $b_i \leq \frac{1}{2}$ otherwise. Since the coefficient $u^{0,0,\ldots,0} = 0$ by hypothesis, the largest $b_{i_1} + \cdots + b_{i_n}$ can be $n - \frac{1}{2}$. This proves our claim. $\square$

As a corollary to this lemma, we have that $\bar{L}_K$ is a contraction on $L^2(\Gamma_{n+\mathcal{N}})$. Let $\int u(v)\Gamma_n(v)dv = 0$, and let Q_R act trivially on u. Then we have

$$|| \left(n\lambda_S Q_S + \mathcal{N}\lambda_R Q_R + n\mu Q_M\right) u||_{L^2(\Gamma_n)} \leq (\Lambda - \frac{\mu}{2})||u||_{L^2(\Gamma_n)}. \tag{4.14}$$

Here

$$\Lambda = n(\lambda_S + \mu) + \mathcal{N}\lambda_R \tag{4.15}$$

We now give the d_2 analog of Propositions 8 and 9. The Maxwellian thermostat has a nice representation in Fourier space. In fact, for $f \in L^1(\mathbb{R})$ we have:

$$\begin{aligned}
\mathcal{F}\{M[h]\} &= \int_{\mathbb{R}}\int_{\mathbb{R}}\fint_0^{2\pi} \exp(-2\pi i v\xi) f(v\cos\theta - w\sin\theta)\Gamma_1(v\sin\theta + w\cos\theta)\, d\theta\, dv\, dw \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\fint_0^{2\pi} \exp(-2\pi i(v'\cos\theta + w'\sin\theta)\xi) f(v')\Gamma_1(w')\, d\theta\, dv\, dw \\
&= \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(\xi\cos\theta)\hat{\Gamma}_1(\xi\sin\theta)\, d\theta =: \hat{M}\left[\hat{h}\right]
\end{aligned}$$

Proposition 10 *Let f be a probability density on $\mathbb{R}^n$. Then*

$$d_2(Q_M f, \Gamma_n) \leq \left(1 - \frac{1}{2n}\right) d_2(f, \Gamma_n).$$

Proof. Recall that for any θ we have $\Gamma_1(x) = \Gamma_1(x\cos\theta)\Gamma_1(x\sin\theta)$. Thus,

$$\begin{aligned}
d_2(Q_M f, \Gamma) &= \sup_{|\xi|\neq 0} |\xi|^{-2} \left| \frac{1}{n}\sum_{i=1}^n \fint_0^{2\pi} (\hat{f} - \hat{\Gamma_n})(\xi_1, \ldots, \xi_i\cos\theta, \ldots, \xi_n)\Gamma_1(\xi_i\sin\theta)\, d\theta \right| \\
&\leq \sup_{|\xi|\neq 0} |\xi|^{-2}\frac{1}{n}\sum_{i=1}^n d_2(f, \Gamma)\left[|\xi|^2 - \frac{\xi_i^2}{2}\right] \\
&= d_2(f, \Gamma)\sup_{|\xi|\neq 0} \frac{n|\xi|^2 - \frac{1}{2}|\xi|^2}{|\xi|^2} \\
&= \left(1 - \frac{1}{2n}\right) d_2(f, \Gamma).
\end{aligned}$$

$\square$

As a corollary, for $f \geq 0$ a probability density on $L^1(\mathbb{R}^n)$, we have

$$d_2\left(\frac{1}{\Lambda}(n\lambda_S Q_S + \mathcal{N}\lambda_R Q_R + n\mu Q_M)f(v)\Gamma_{\mathcal{N}}(w), \Gamma_{n+\mathcal{N}}(v,w)\right) \leq \left(1 - \frac{\mu}{2\Lambda}\right) d_2(f, \Gamma_{\mathcal{N}}). \tag{4.16}$$

From the convexity property (3.3) of d_2, we also have that

$$d_2\left(\frac{1}{\Lambda}(n\lambda_S Q_S + n\mu Q_I + \mathcal{N}\lambda_R Q_R)f(v)\Gamma_{\mathcal{N}}(w), \Gamma_{n+\mathcal{N}}(v,w)\right) \leq d_2(f(v)\Gamma_{\mathcal{N}}(w), \Gamma_{n+\mathcal{N}}(v,w)) \tag{4.17}$$

where we used $\frac{1}{\Lambda}(n\lambda_S Q_S + n\mu Q_I + \mathcal{N}\lambda_R Q_R)[\Gamma_{n+\mathcal{N}}(v,w)] = \Gamma_{n+\mathcal{N}}(v,w)$.

The proof of Theorems 6 and 7 and is based on the expansion of the exponential in a Taylor series. Let A and B be bounded operators. Then, for any $k \geq 1$ we have

$$A^k - B^k = \sum_{j=0}^{k-1} A^j(A-B)B^{k-j-1}.$$

It follows that

$$e^{tA} - e^{tB} = \sum_{k=1}^{\infty} \frac{t^k}{k!} \sum_{j=0}^{k-1} A^j(A-B)B^{k-j-1}.$$

Letting $A = \Lambda I - L_{FR} = \mathcal{N}\lambda_R Q_R + n(\lambda_S Q_S + \mu Q_I)$ and $B = \Lambda I - L_T = \mathcal{N}\lambda_R Q_R + n(\lambda_S Q_S + \mu Q_M)$, we see that

$$e^{-tL_{FR}}f - e^{-tL_T}f = e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{\Lambda^k t^k}{k!}\sum_{j=0}^{k-1}\left(\frac{\mathcal{N}\lambda_R Q_R + n\lambda_S Q_S + n\mu Q_I}{\Lambda}\right)^j \frac{n\mu}{\Lambda}(Q_I - Q_M) \\ \times \left(\frac{\mathcal{N}\lambda_R Q_R + n(\lambda_S Q_S + \mu Q_M)}{\Lambda}\right)^{k-j-1} f.$$

This expansion has some advantages listed below:

1. The factor $e^{-\Lambda t}$ avoids expanding a negative exponential in a Taylor series.
2. Each summand contains the factor $(Q_I - Q_M)$ which we will show in Lemmas 6 for the L^2 case and 11 for the d_2 case that it gets smaller with $\mathcal{N}$ provided it acts on functions of the form $\phi(v)\Gamma_{\mathcal{N}}(w)$.

3. If $f(v,w) = l_0(v)\Gamma_{\mathcal{N}}(w)$, then $(\frac{\mathcal{N}\lambda_R Q_R + n(\lambda_S Q_S + \mu Q_M)}{\Lambda})^{k-j-1} f$ is a probability density that factors in a similar way since Q_R only acts on the w variables while Q_S and Q_M act only on the v variables.

4. The first factor: A^j is an averaging operator, and $||A^j u||_2 \le ||u||_2$. Equality holds only when $u \equiv 1$. We will show in the proof of Theorem 6 that A^j is going to act on functions u with $\int u\Gamma_{n+\mathcal{N}} = 0$. Thus, Proposition 9 can be used to deduce $||A^j u||_2 \le \left(\Lambda - \frac{\mu}{2}\right)^j ||u||_2$. Similarly, equation (4.17) implies that $d_2(A^k\phi, \Gamma_{n+\mathcal{N}}) \le \left(\Lambda - \frac{\mu}{2}\right)^k d_2(\phi, \Gamma_{n+\mathcal{N}})$.

We see that the heart of the proof of both theorems lies in comparing Q_I and Q_M (or Q_I and $Q_{\bar{M}}$). We first do this for the L^2 case in following section.

4.1 $L_{FR} - L_T$ Comparison in L^2

Lemma 6 *Let h be in $L^2(\Gamma_n)$. Then h can be seen as a function in $L^2(\Gamma_{n+\mathcal{N}})$ and*

$$||(Q_{\bar{M}} - Q_I)[h]||_{L^2(\mathbb{R}^{n+\mathcal{N}}, \Gamma_{n+\mathcal{N}})} \le \frac{1}{2\sqrt{\mathcal{N}}} ||h||_{L^2(\mathbb{R}^n, \Gamma_n)} \tag{4.18}$$

Proof.- We first study the claim for the case $n = 1$. $f(v,w) = h(v_1)\Gamma_{1+\mathcal{N},\beta}(v,w)$.

$$\begin{aligned} ||(Q_{\bar{M}} - Q_I)h(v_1)||^2_{L^2} &= \int \Gamma(v_1,w)dv_1dw \left[\bar{M}_1(h)^2 - 2\bar{M}_1 h \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \int h(v_1\cos\theta - w_j \sin\theta)d\theta + \right. \\ &\left. \frac{1}{\mathcal{N}^2} \sum_{k,l} \int\!\!\int h(v_1\cos\theta_1 - w_k\sin\theta_1)h(v_1\cos\theta_2 - w_l\sin\theta_2)d\theta_1\, d\theta_2 \right]. \end{aligned}$$

We can integrate with respect to w_j first in the first line. In the second line, we integrate first with respect to w_j, w_k when $j \ne k$. Noting that $\int \Gamma(w_j)\, dw_j = 1$, we get

$$\begin{aligned} ||(Q_{\bar{M}} - Q_I)h(v_1)||^2_{L^2} &= \int \Gamma(v_1,w)dv_1dw \left[(\bar{M}_1 h)^2 - 2(\bar{M}_1 h)^2 + \frac{\mathcal{N}^2 - \mathcal{N}}{\mathcal{N}^2}(\bar{M}_1 h)^2 + \right. \\ &\left. \frac{1}{\mathcal{N}^2} \sum_{k} \int\!\!\int h(v_1\cos\theta_1 - w_k\sin\theta_1)h(v_1\cos\theta_2 - w_k\sin\theta_2)d\theta_1\, d\theta_2 \right]. \end{aligned}$$

$$= \int \Gamma(v_1,w) \left[-\frac{1}{\mathcal{N}}(\bar{M}_1 h)^2 + \frac{1}{\mathcal{N}} \int\!\!\int d\theta_1\, d\theta_2 h(v\cos\theta_1 - w_1\sin\theta_1)h(v\sin\theta_2 - w_1\sin\theta_2) \right] dv_1\, dw.$$

The last integral simplifies after we introduce the change of variables: $v^* = v\cos\theta_1 - w\sin\theta_1$ and $w^* = v\sin\theta_1 + w\cos\theta_1$. So $v\cos\theta_2 - w\sin\theta_2$ becomes $v^*\cos(\theta_2-\theta_1) - w^*\sin(\theta_2-\theta_1)$. We have:

$$\int \Gamma(v_1, w) dv_1 dw \frac{1}{\mathcal{N}} \fint\fint d\theta_1\, d\theta_2 h(v^*) h(v^*\sin(\theta_2-\theta_1) - w^*\sin(\theta_2-\theta_1)),$$

where we used the rotational invariance of the Gaussian: $\Gamma(v_1^*, w^*) = \Gamma(v_1, w)$. This integral is $\int\, dv_1 \Gamma(v_1)\, dv_1 h(v) \bar{M}_1 h(v) dv$. Therefore

$$||(Q_{\bar{M}} - Q_I) h(v_1)||_{L^2(\Gamma)} = \frac{1}{\sqrt{\mathcal{N}}} \left(\int \Gamma(v) [h(v) \bar{M}_1 h(v) - (\bar{M}_1 h)^2]\, dv \right)^{\frac{1}{2}}. \tag{4.19}$$

It remains to compare the right-hand-side of (4.19) to $||h||_{L^2(\Gamma)}$. We use the Hermite polynomial expansion. Recall from equation (2.5) that $\bar{M}_1 H_{2k} = \fint \cos(\theta)^{2k}\, d\theta\, H_{2k}$, while $\bar{M}\phi = 0$ when ϕ is odd.

Let $h(v) = \sum c_i H_{2i}(v)$. Then

$$\begin{aligned} \int \bar{M}_1 h (h - \bar{M}_1 h)\, \Gamma(v)\, dv &= \sum_i (1-\lambda_i)\lambda_i |c_i|^2 ||\, H_{2i}||_{L^2}^2 \\ &\leq \frac{1}{4} \sum_i |c_i|^2 ||\, H_{2i}||_{L^2}^2 = \frac{1}{4} ||h||^2 \end{aligned}$$

This gives us

$$||(Q_{\bar{M}} - Q_I) h||_{L^2(\Gamma)} \leq \frac{1}{2\sqrt{\mathcal{N}}} ||h||_{L^2(\Gamma)} \tag{4.20}$$

Showing us a gain of a factor $\frac{1}{\sqrt{\mathcal{N}}}$.

If u depends on n variables, the computation leading to (4.19) leads to

$$||\bar{M}_i u - \frac{1}{\mathcal{N}} \sum_{j=1}^{n} Q_{i,j}^I u|| = \frac{1}{\sqrt{\mathcal{N}}} \left(\int \Gamma(v) \bar{M}_i u (u - \bar{M}_i u)\, dv \right)^{\frac{1}{2}} \leq \frac{1}{2\sqrt{\mathcal{N}}} ||u||_{L^2(\Gamma)} \tag{4.21}$$

The inequality is obtained after integrating with respect to the variable v_i first, and using the steps before (4.20).

The L^2 norm of the full $(Q_{\bar{M}} - Q_I)u$ is

$$\begin{aligned}
||(Q_{\bar{M}} - Q_I)u||_{L^2} &= \left\| \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}\left(\bar{M}_i u - Q^I_{i,j}u\right)\right\|_{L^2(\Gamma)} \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left\|\frac{1}{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}\left(\bar{M}_i u - Q^I_{i,j}u\right)\right\|_{L^2(\Gamma)} \\
&\leq \frac{1}{2\sqrt{\mathcal{N}}}||u||_{L^2(\Gamma)}
\end{aligned}$$

□

From this, the proof of Theorem 6 quickly follows. We replace f by $h\Gamma_{n+\mathcal{N}}$ in equation (5.3) and obtain

$$||\left(e^{-tL_{FR}} - e^{-t\bar{L}_T}\right)h||_{L^2(\Gamma)} \leq \frac{n\mu}{\Lambda}e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{(\Lambda t)^k}{k!}\sum_{j=0}^{k-1}\frac{1}{2\sqrt{\mathcal{N}}}\left\|\frac{\bar{L}_T^{k-j-1}}{\Lambda^{k-j-1}}[h-1]\right\|_{L^2(\Gamma)} \tag{4.22}$$

Here the term 1 appeared because $e^{-tL_{FR}}1 - e^{-t\bar{L}_T}1 \equiv 0$. Thus,

$$\begin{aligned}
||\left(e^{-tL_{FR}} - e^{-t\bar{L}_T}\right)h||_{L^2(\Gamma)} &\leq \frac{1}{2\sqrt{\mathcal{N}}}\frac{n\mu}{\Lambda}e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{(\Lambda t)^k}{k!}\sum_{j=0}^{k-1}(1-\frac{\mu}{2\Lambda})^{k-j-1}\,||h-1||_{L^2(\Gamma)} \\
&= \frac{1}{2\sqrt{\mathcal{N}}}\frac{n\mu}{\Lambda}e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{(\Lambda t)^k}{k!}\left(1-(1-\frac{\mu}{2\Lambda})^k\right)\frac{2\Lambda}{\mu}\,||h-1||_{L^2(\Gamma)} \\
&= \frac{n}{\sqrt{\mathcal{N}}}\left(1-e^{-\frac{\mu}{2}t}\right)||h-1||_{L^2(\Gamma)};
\end{aligned}$$

as desired. □

At $t=0$, both sides of this inequality are zero. So this inequality is exact near $t=0$. The right-hand side goes to zero as $\mathcal{N}^{-\frac{1}{2}}$ uniformly in time. This is the same rate as in the infinite time limit (see Appendix B). The behavior in n is not the same as the behavior in n in the infinite time limit. The proof of Theorem 6 is based on Lemma 6 which is exact at $t=0$ as shown in Appendix C. It is not known if the distance of both evolutions actually reaches a distance of order $\frac{n}{\mathcal{N}}$ in L^2. Luckily, the extra factor $\sqrt{n}$ is negligible compared to $||h-1||_{L^2(\Gamma)}$ which can be of order $C^{\mathcal{N}}$.

4.2 $L_{FR} - L_T$ comparison in d_2

The proof of Theorem 11 below is much more involved. We want to study

$$d_2(Q_I[\phi(v)\Gamma_{\mathcal{N}}(w)], Q_M[\phi(v)]\Gamma_{\mathcal{N}}(w)). \tag{4.23}$$

for general ϕ with finite 4^{th} moment, and to bound it in terms of $d_2(\phi, \Gamma_n)$. We smoothen the proof of Proposition 11 by rewriting (4.23) in an explicit form that will be useful for computations. We have

$$d_2(Q_I[\phi\Gamma_{\mathcal{N}}], Q_M[\phi\Gamma_{\mathcal{N}}]) = \frac{1}{n\mathcal{N}} \sup_{\vec{\xi},\vec{\eta}\neq 0} \frac{1}{|\vec{\xi}|^2 + |\vec{\eta}|^2} \left| \sum_{i=1}^{n} \sum_{j=1}^{\mathcal{N}} \left(\widehat{Q}^I_{i,j}[\widehat{\phi}\Gamma_{\mathcal{N}}] - \widehat{M}_i[\widehat{\phi}\Gamma_{\mathcal{N}}] \right) \right|. \tag{4.24}$$

First we notice that a factor of $\Gamma_{\mathcal{N}-1}$ can be pulled out.

$$\begin{aligned}\widehat{Q}^I_{i,j}[\widehat{\phi}\Gamma_{\mathcal{N}}] =& \Gamma_{\mathcal{N}-1}(\eta^j) \fint d\theta \hat{\phi}(\xi_1, \ldots, \xi_i \cos\theta + \eta_j \sin\theta, \ldots, \xi_n)\Gamma_1(-\xi_i \sin\theta + \eta_j \cos\theta) := \\ :=& \widehat{F}_i(\vec{\xi}, \eta_j)\Gamma_{\mathcal{N}-1}(\eta^j)\end{aligned}$$

where

$$\vec{\eta}^j = (\eta_1, \ldots, \eta_{j-1}, \eta_{j+1}, \ldots, \eta_{\mathcal{N}}), \tag{4.25}$$

while

$$\widehat{M}_i[\widehat{\phi}\Gamma_{\mathcal{N}}] = \Gamma_{\mathcal{N}}(\eta) \fint d\theta \hat{\phi}(\xi_1, \ldots, \xi_i \cos\theta, \ldots, \xi_n)\Gamma_1(-\xi_i \sin\theta) = \widehat{F}_i(\vec{\xi}, 0)\Gamma_1(\eta_j)\Gamma_{\mathcal{N}-1}(\eta^j).$$

Thus calling

$$\widehat{G}(\vec{\xi}, \eta) = \sum_{i=1}^{n} \left(\widehat{F}_i(\vec{\xi}, \eta) - \widehat{F}_i(\vec{\xi}, 0)\Gamma_1(\eta) \right) \tag{4.26}$$

we get

$$d_2(Q_I[\phi\Gamma_{\mathcal{N}}], Q_M[\phi\Gamma_{\mathcal{N}}]) = \frac{1}{n\mathcal{N}} \sup_{\vec{\xi},\vec{\eta}\neq 0} \frac{1}{|\vec{\xi}|^2 + |\vec{\eta}|^2} \left| \sum_{j=1}^{\mathcal{N}} \widehat{G}(\vec{\xi}, \eta_j)\Gamma_{\mathcal{N}-1}(\vec{\eta}^j) \right|. \tag{4.27}$$

Clearly $\hat{G}(\vec{\xi}, 0) = 0$. We show next that $\hat{G}(\vec{\xi}, \eta)$ is even in η. This is because each F_i, and thus

$\hat{F}_i$, is even in its second variable for

$$F_i(v,w) = \int_0^{2\pi} d\theta \hat{\phi}(v_1,\ldots,v_i\cos\theta + w\sin\theta,\ldots,v_n)\Gamma_1(-v_i\sin\theta + w\cos\theta) =$$
$$= \int_0^{2\pi} d\theta \hat{\phi}(v_1,\ldots,v_i\cos(-\theta) - w\sin(-\theta),\ldots,v_n)\Gamma_1(v_i\sin(-\theta) - w\cos(-\theta)) = F_i(v,-w)$$

here we have used the evenness of Γ_1.

We are going to gain from the supremum over the $\vec{\eta}$ variables of the reservoirs. Let

$$\mathcal{D}_{\mathcal{N}}\left(H(.),a\right) = \sup_{\vec{\eta}\neq 0}\left|\frac{1}{a^2+|\vec{\eta}|^2}\sum_{j=1}^{\mathcal{N}} H(\eta_j)\Gamma_{\mathcal{N}-1}(\vec{\eta}^j)\right|. \tag{4.28}$$

Then

$$d_2(Q_I[\phi\Gamma_{\mathcal{N}}],Q_M[\phi\Gamma_{\mathcal{N}}]) = \frac{1}{n\mathcal{N}}\sup_{\vec{a}\neq 0}\mathcal{D}_{\mathcal{N}}(H_{\vec{a}}(.),|a|),$$

with $H_{\vec{a}}(\eta) = \hat{G}(\vec{a},\eta)$.

We will study $\mathcal{D}_{\mathcal{N}}\left(H_{\vec{\xi}}(.),|\vec{\xi}|\right)$ and show in the next proposition that it can be bounded in terms of $\mathcal{D}_1\left(H_{\vec{\xi}}(.),|\vec{\xi}|\right)$ and $\max_{p\leq 4}|\partial_\eta^p H_{\vec{\xi}}(\eta)|_{L^\infty(\mathbb{R})}$. This reduces the supremum over $\vec{\eta}\in\mathbb{R}^{\mathcal{N}}$ to a supremum over $\eta\in\mathbb{R}$. Now we are ready for the statement and proof of Prop 11.

Proposition 11 *Let $H(\eta)$ be a bounded C^4 function of η. Assume that*

$$H(0) = 0 \qquad\qquad H(\eta) = H(-\eta)$$

and

$$C_4 = \|H(\cdot)\|_{C^4} := \max_{p\leq 4}\sup_{\eta}\left|\frac{d^p}{d\eta^p}H(\eta)\right| < \infty. \tag{4.29}$$

Let $\mathcal{D}_{\mathcal{N}}(H,a)$ be as in (4.28)*. Then*

$$\mathcal{D}_{\mathcal{N}}(H,a) \leq \left[(8C_4 + \mathcal{D}_1(H,a))\mathcal{D}_1(H,a)\right]^{\frac{1}{2}} \tag{4.30}$$

Equation (4.30) cannot be improved to an equation of the form $\mathcal{D}_{\mathcal{N}}(H,a) \leq K\mathcal{D}_1(H,a)$ with K independent of $\mathcal{N}$. This is shown in Appendix D.

It is not surprising that $\mathcal{D}_{\mathcal{N}}(H,a)$ is of order 1. That is because

$$\mathcal{D}_{\mathcal{N}}(H,a) \leq \sup_{\vec{\eta}\neq\vec{0}} \frac{|H(a,\eta_j)\Gamma(\eta^j)|}{|\vec{\eta}|^2} = \sup_{x\neq 0} \frac{|H(a,x)|}{x^2} < \infty.$$

This estimate is not strong enough to give information about $d_2(Q_I[\phi\Gamma_{\mathcal{N}}], Q_M[\phi\Gamma_{\mathcal{N}})$ since it leads only to the bound

$$d_2(Q_I[\phi\Gamma_{\mathcal{N}}], Q_M[\phi\Gamma_{\mathcal{N}}]) \leq \frac{1}{n\mathcal{N}} \sup_{(\vec{\xi},\vec{\eta})\neq\vec{0}} \frac{1}{|\vec{\eta}|^2} \left| \sum_{j=1}^{\mathcal{N}} \left(Q^I_{i,j}[\hat{l}_k\Gamma_1](\vec{\xi},\eta) - \hat{M}_i[\hat{l}_k\Gamma_1](\vec{\xi},\eta) \right) \right|.$$

Without the $|\vec{\xi}|^2$ in the denominator, one cannot directly compare it to $d_2(l_0,\Gamma_n)$. [2] The proof will retain information from a in the denominator of $\mathcal{D}_{\mathcal{N}}(H,a)$ using the opposing forces: the condition $H(0)=0$ that drives $\vec{\eta}$ away from zero in the supremum in $\mathcal{D}_N(H,a)$, and the finite fourth moment condition which stops $|\vec{\eta}|$ from growing too large in the supremum in $\mathcal{D}_N(H,a)$.

Proof .- The inequality (4.30) is obtained in 2 stages. The first stage constructs the upper bound (4.35) for $\mathcal{D}_{\mathcal{N}}(H,a)$ in terms of $\mathcal{D}_{\mathcal{N}}(H,a), \mathcal{D}_{\mathcal{N}}(H,0)$, and a using an upper bound $\widetilde{H}$ for $|H|$. $\tilde{H}$ retains the quadratic behavior of H near the origin. The second stage analyzes (4.35) and connects it to $\mathcal{D}_1$ and to C_4 in (4.39) using a Taylor expansion of $H(\eta)$.

From (4.28) it follows that

$$|H(\eta)| \leq \mathcal{D}_1(H,a)(\eta^2+a^2). \tag{4.31}$$

The following definition serves to simplify the upper bound in (4.28) by including only the quadratic behavior of $H(\vec{\xi})$ near the origin. Let

$$\widetilde{H}(\eta,a) = \min\{\mathcal{D}_1(H,0)\eta^2, \mathcal{D}_1(H,a)(a^2+\eta^2)\} = \begin{cases} \mathcal{D}_1(H,0)\eta^2 & \eta^2 \leq \eta_0^2(a) \\ \mathcal{D}_1(H,a)(a^2+\eta^2) & \eta^2 \geq \eta_0^2(a) \end{cases}$$

where

$$\eta_0^2(a) = \frac{\mathcal{D}_1(H,a)a^2}{\mathcal{D}_1(H,0)-\mathcal{D}_1(H,a)} \tag{4.32}$$

[2]When $n=1$, it is possible to arrive at a comparision result using a Fourier metric without $|\vec{\xi}|^2$ in the denominator. In an early stage of our work on Theorem 7 we used $d_1(h)$ and $\mathcal{D}(h) = \sup_{\vec{\xi}\neq\vec{0}} \frac{|\hat{h}(\sqrt{\xi^2+\eta^2})-\hat{h}(\xi)|}{|\vec{\eta}|^2}$ for $h\in L^1(\mathbb{R})$ even, with finite second moment, and centered at the origin. Definition 3.1.2 and Theorem 3.1.12 in Ranjini Vaidyanathan's thesis [27] contain the interesting details.

is chosen so that $\widetilde{H}$ continuous. We get $|H(\eta)| \leq \widetilde{H}(\eta, a)$ and thus $\mathcal{D}_{\mathcal{N}}(H, a) \leq \mathcal{D}_{\mathcal{N}}(\widetilde{H}, a)$.

We now analyze $\mathcal{D}_{\mathcal{N}}(H, a)$.

Lemma 1 *Under the hypotheses of Proposition 11 we have*

$$\mathcal{D}_{\mathcal{N}}(\widetilde{H}, a) = \mathcal{D}_1(H, 0) \sup_{k \leq \mathcal{N}, |\eta| \leq \eta_0(a)} \frac{k\eta_0(a)^2 e^{-\pi((k-1)\eta_0(a)^2+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{a^2 + k\eta_0(a)^2 + \eta^2} \tag{4.33}$$

that is, the supremum in (4.28) *for $\widetilde{H}$ is attained for $\vec{\eta}$ of the form $\vec{\eta} = (\eta_0(a), \ldots, \eta_0(a), \eta, 0, \ldots, 0)$ for some η with $|\eta| \leq \eta_0(a)$.*

Proof (of Lemma 1).- Let

$$\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) = \frac{\sum_{i=1}^{\mathcal{N}} \widetilde{H}(\eta_i)\Gamma_{\mathcal{N}-1}(\vec{\eta}^i)}{a^2 + |\vec{\eta}|^2}$$

and suppose $\vec{\eta}$ has $|\eta_i| > \eta_0(a)$ for some i. By differentiating we get

$$\partial_{\eta_i}\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) = \partial_{\eta_i}\left(\widetilde{H}(\eta_i)e^{\pi\eta_i^2}\right)\frac{\Gamma_{\mathcal{N}}(\vec{\eta})}{a^2 + \vec{\eta}^2} - 2\eta_i\left(\pi + \frac{1}{a^2 + \vec{\eta}^2}\right)\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}).$$

Now we use

$$\partial_\eta\left(\widetilde{H}(\eta)e^{\pi\eta^2}\right) = 2\eta\left(\pi\widetilde{H}(\eta) + \mathcal{D}_1(H, a)\right)e^{\pi\eta^2}$$

whenever $\eta \geq \eta_0(a)$. This results in

$$\partial_{\eta_i}\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) = 2\eta_i\pi\left(\widetilde{H}(\eta_i)\frac{\Gamma_{\mathcal{N}-1}(\vec{\eta}^i)}{a^2 + |\vec{\eta}|^2} - \widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta})\right) + 2\eta_i\left(\frac{\mathcal{D}_1(H, a)\Gamma_{\mathcal{N}-1}(\vec{\eta}^i) - \mathcal{H}_{\mathcal{N}}(a, \vec{\eta})}{a^2 + |\vec{\eta}|^2}\right)$$

Both bracketed terms with coefficients $2\eta_i\pi$ and $2\eta_i$ respectively are negative, with equality holding if and only if $\vec{\eta}^i = \vec{0}$. Thus

$$\partial_{\eta_i}\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) < 0, \text{ when } \eta_i > \eta_0.$$

This implies that

$$\sup_{\vec{\eta} \neq 0} \widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) = \sup_{\vec{\eta} \neq 0, |\eta_i| \leq \eta_0} \widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta})\,.$$

It remains to show that there can be at most 1 coordinate i such that $0 < |\eta_i| < \eta_0(a)$. If

$$L(x, y) := x^2 e^{\pi x^2} + y^2 e^{\pi y^2},$$

then $L(r\cos\theta, r\sin\theta)$ is maximal for $\theta = k\frac{\pi}{2}$ and minimal for $\theta = \frac{\pi}{4} + k\frac{\pi}{2}$. Moreover, it is strictly increasing for $\frac{\pi}{4} + k\frac{\pi}{2} < \theta < (k+1)\frac{\pi}{2}$ and strictly decreasing for $k\frac{\pi}{2} < \theta < k\frac{\pi}{2} + \frac{\pi}{4}$.

So if $0 < \eta_1 < \eta_0(a)$ and $0 < \eta_2 < \eta_0(a)$, then $L(\eta_1, \eta_2) \leq L(x_0 = \max\left\{\eta_0(a), \sqrt{\eta_1^2 + \eta_2^2}\right\}, \sqrt{\eta_1^2 + \eta_2^2 - x_0^2})$. For $|\eta_i| \leq \eta_0(a)$ we have

$$\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta}) = \frac{B(a)L(\eta_1, \eta_2)\Gamma_{\mathcal{N}-2}(\eta_3 \ldots, \eta_{\mathcal{N}}) + \sum_{i=3}^{\mathcal{N}} \widetilde{H}(a, \eta_i)\Gamma_{\mathcal{N}-1}(\vec{\eta}^i)}{a^2 + |\vec{\eta}|^2},$$

so that there can be no maximum for $\widetilde{\mathcal{H}}_{\mathcal{N}}(a, \vec{\eta})$ for which both $0 < \eta_1 < \eta_0(a)$ and $0 < \eta_2 < \eta_0(a)$. Repeating this argument for each pair η_i, η_j with $1 \leq i, j \leq \mathcal{N}$ shows that for all, but possibly one, i we must have $\eta_i = 0$ or $\eta_i = \eta_0(a)$.

□

To complete the proof of the first part of Proposition 11 we will simplify the right-hand side of equation (4.33). Observe first that

$$\frac{k\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{a^2 + k\eta_0(a)^2 + \eta^2} \leq \max\left\{\frac{\eta_0^2(a)}{\frac{a^2}{2} + \eta_0(a)^2}, \frac{(k-1)\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2}\right\}.$$

This follows from the inequality $\frac{a+c}{b+d} \leq \max\{\frac{a}{b}, \frac{c}{d}\}$ for a, b, c, and d positive numbers; as shown in the proof of (3.6).

From (4.32) we have

$$\frac{\eta_0^2(a)}{\frac{a^2}{2} + \eta_0(a)^2} \leq 2\frac{\mathcal{D}_1(H, a)}{\mathcal{D}_1(H, 0)}$$

while

$$\sup_{k\leq\mathcal{N}, |\eta|\leq\eta_0(a)} \frac{(k-1)\eta_0(a)^2 e^{-\pi((k-1)\eta_0^2(a)+\eta^2)} + \eta^2 e^{-\pi k\eta_0(a)^2}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2} \leq \sup_{k\leq\mathcal{N}, |\eta|\leq\eta_0(a)} \frac{((k-1)\eta_0(a)^2 + \eta^2) e^{-\pi((k-1)\eta_0^2(a)+\eta^2)}}{\frac{a^2}{2} + (k-1)\eta_0(a)^2 + \eta^2} \leq 2\sup_{y>0} \frac{ye^{-\pi y}}{\frac{a^2}{2} + y}. \tag{4.34}$$

Clearly we have

$$\frac{ye^{-\pi y}}{\frac{a^2}{2} + y} \leq \frac{y}{(\frac{a^2}{2} + y)(1 + \pi y)} \leq \frac{1}{\frac{\pi a^2}{2} + 1}$$

so that

$$\mathcal{D}_{\mathcal{N}}(H,a) \leq \max\left\{\mathcal{D}_1(H,a), 2\frac{\mathcal{D}_1(H,0)}{1+\frac{\pi}{2}a^2}\right\}. \tag{4.35}$$

This concludes the first stage of the proof.

The second stage consists of replacing $\mathcal{D}_1(H,0)$ by expressions involving $\mathcal{D}_1(H,a)$ and C_4. From the hypotheses of Proposition 11, it follows that

$$\frac{|H''(0)|\eta^2}{2} - \frac{C_4\eta^4}{4!} \leq |H(\eta)| \leq \frac{|H''(0)|\eta^2}{2} + \frac{C_4\eta^4}{4!}. \tag{4.36}$$

Let $M = \sup_\eta |H(\eta)|$. Since $H(0) = 0$, by continuity there exists a finite $\tilde{\eta} > 0$ such that $|H(\tilde{\eta})| > M/2$. This shows that $\mathcal{D}_1(H,0) \geq M/(2\tilde{\eta}^2)$. while

$$\frac{|H(\eta)|}{\eta^2} < \frac{M}{2\tilde{\eta}^2} \qquad \text{if} \qquad \eta^2 > 2\tilde{\eta}^2$$

Thus there exists η_* such that $\eta_*^2 \leq 2\tilde{\eta}^2$ and $|H(\eta_*)| = \mathcal{D}_1(H,0)\eta_*^2$. We also know from (4.31) that

$$|H''(0)| \leq 2\mathcal{D}_1(H,0),$$

with equality if and only if $\eta_*^2 = 0$. We will use these observations to show the following relation:

Lemma 2 *Under the hypotheses of Proposition 11 we have*

$$\mathcal{D}_1(H,a) \geq \frac{\mathcal{D}_1(H,0)^2}{\frac{3}{2}C_4a^2 + 4\mathcal{D}_1(H,0)}$$

Proof (of the lemma).- From (4.36) it follows that

$$\frac{|H(a,\eta)|}{a^2+\eta^2} \geq \frac{\frac{|H''(0)|\eta^2}{2} - \frac{C_4\eta^4}{4!}}{a^2+\eta^2}$$

and, choosing η^2 to be $\frac{6|H''(0)|}{C_4}$, we get that

$$\sup_\eta \frac{|H(a,\eta)|}{a^2+\eta^2} \geq \frac{|H''(0)|^2}{4|H''(0)| + \frac{3}{2}C_4a^2}. \tag{4.37}$$

Since there is no positive lower bound for $|H''(0)|$, we use the second inequality in (4.36) to complement inequality (4.37). We find that for all η

$$|H(\eta)| - \mathcal{D}_1(H,0)\eta^2 \leq \frac{(|H''(0)| - 2\mathcal{D}_1(H,0))\eta^2}{2} + \frac{C_4\eta^4}{4!}.$$

Since $|H''(0)| - 2\mathcal{D}_1(H,0) \leq 0$ we get

$$\eta_*^2 \geq \frac{12(2\mathcal{D}_1(H,0) - |H''(0)|)}{C_4}.$$

Here η_* could be zero as mentioned above.

This implies that

$$\sup_\eta \frac{|H(\eta)|}{a^2+\eta^2} \geq \frac{|H(\eta_*)|}{a^2+\eta_*^2} \geq \liminf_{\epsilon\to 0} \frac{|H(\eta_*)|}{\eta_*^2+\epsilon}\frac{\eta_*^2}{a^2+\eta_*^2} \geq \frac{12\mathcal{D}_1(H,0)(2\mathcal{D}_1(H,0) - |H''(0)|)}{C_4a^2 + 12(2\mathcal{D}_1(H,0) - |H''(0)|)}\,. \tag{4.38}$$

Observe now that the right-hand side of (4.37) is an increasing function of $|H''(0)|$ while the right-hand side of (4.38) is decreasing. Thus, for every $0 \leq h \leq 2\mathcal{D}_1(H,0)$ we have

$$\mathcal{D}_1(H,a) \geq \min_{0\leq h\leq 2\mathcal{D}_1(H,0)} \max\left\{\frac{h^2}{4h+\frac{3}{2}C_4a^2}\,,\,\frac{12\mathcal{D}_1(H,0)(2\mathcal{D}_1(H,0)-h)}{C_4a^2+12(2\mathcal{D}_1(H,0)-h)}\right\}$$

For $h \leq \mathcal{D}_1(H,0)$, $\frac{12\mathcal{D}_1(H,0)(2\mathcal{D}_1(H,0)-h)}{C_4a^2+12(2\mathcal{D}_1(H,0)-h)} \geq \frac{12\mathcal{D}_1(H,0)^2}{12\mathcal{D}_1(H,0)^2+C_4a^2}$, while for $h \geq \mathcal{D}_1(H,0)$, $\frac{h^2}{4h+\frac{3}{2}C_4a^2} \geq \frac{\mathcal{D}_1(H,0)^2}{\frac{3}{2}C_4a^2+4\mathcal{D}_1(H,0)}$. The proof of Lemma 2 now follows from the observation that $\frac{\mathcal{D}_1(H,0)^2}{\frac{3}{2}C_4a^2+4\mathcal{D}_1(H,0)} \leq \frac{12\mathcal{D}_1(H,0)^2}{12\mathcal{D}_1(H,0)^2+C_4a^2}$. $\square$

Observe finally that $2\mathcal{D}_1(H,0) \leq \sup_\eta |H''(\eta)| \leq C_4$ since $2|H(\eta)|/\eta^2 \leq \sup_\eta |H''(\eta)|$. Thus we can write

$$\mathcal{D}_1(H,a) \geq \frac{2\mathcal{D}_1(H,0)^2}{C_4}\frac{1}{3a^2+4}\,. \tag{4.39}$$

Putting together (4.35) and (4.39) we get the claim. $\square$

We prepare for using Proposition 11 and complete the proof of Theorem 7 as follows. Let $\phi \geq 0$ be in $L^1(\mathbb{R}^n)$, symmetric in its variables, and have finite moments up to order 4. Let $E_\phi = \max_{k\leq 4} \int |v_1|^k\phi(v)\,dv$. Let $H_\xi(\eta) = \hat{G}(\vec{\xi},\eta)$ as suggested in (4.26). We now show that $H_\xi(\eta)$ is has bounded 4^th order derivatives (in η). Observe that for $p \leq 4$ Jensen's inequality gives:

$$\begin{aligned}\left|\partial_\eta^p \widehat{Q}^I_{i,1}[\widehat{l}_k\Gamma_1](\vec{\xi},\vec{\eta})\right| \leq &(2\pi)^4\int |w|^p Q^I_{i,1}[l_k\Gamma_1](v,w)dvdw \leq (2\pi)^4\left(\int |w|^4 Q^I_{i,1}[l_k\Gamma_1](v,w)dvdw\right)^{\frac{p}{4}}\\ =&(2\pi)^4\left(\frac{3}{8}\int (w^2+v_i^2)^2 l_k(v)\Gamma_1(w)dvdw\right)^{\frac{p}{4}} = (2\pi)^4\left(\frac{3}{8}\right)^{\frac{p}{4}}\left(E_{\phi,4}+2\frac{E_{\phi,2}}{\sqrt{2\pi}}+\frac{3}{2\pi}\right)^{\frac{p}{4}}\\ &\leq (2\pi)^4\max\left\{\frac{3}{8}\left(E_\phi+\frac{1}{\pi}\right),1\right\}.\end{aligned}$$

Here $E_{\phi,k} = \int_{\mathbb{R}^n}\phi(v)\,|v|^k\,dv$ and Young's inequality $E_{\phi,2}\leq(1+E_{\phi,4})/2$ was used.

$$\begin{aligned}|\partial_\eta^p H_{\vec{\xi}}(\eta)| &= \left|\partial_\eta^p\sum_{i=1}^n[\widehat{Q}^I_{i,1}-\widehat{M}_i][\widehat{\phi}\Gamma_1](\vec{\xi},\eta)\right|\\ &= (2\pi)^p\left|\int_{\mathbb{R}^{n+1}} w^p\sum_{i=1}^n\big(Q^I_{i,1}[\phi\Gamma_1](v,w)+M_i[\phi\Gamma_1](v,w)\big)\,dv\,dw\right|\\ &\leq (2\pi)^p\sum_{i=1}^n\left[\left|\int_{\mathbb{R}^{n+1}} w^p Q^I_{i,1}[\phi\Gamma_1](v,w)dv\,dw\right|+\int w^p\Gamma_1(w)\,dw\right]\\ &\leq (2\pi)^4 n\left[\max\left\{\frac{3}{8}\left(E_\phi+\frac{1}{\pi}\right),1\right\}+1\right] =: C_4(\phi),\end{aligned} \tag{4.40}$$

where we used the fact that for $\beta = 2\pi$, $\left\{\int|w|^p\Gamma_1(w)\,dw\right\}_{p\geq 0}$ is a decreasing sequence, and thus bounded by 1.

Using (4.26) we get

$$\|\widehat{G}_k(\vec{\xi},\cdot)\|_{C^4}\leq 12\pi^4\left(E_{4,k}+\frac{1}{\pi}\right), \tag{4.41}$$

where

$$E_{j,k} = \int |v_1|^j l_k(v)dv = \int |v_1|^j\Lambda^{-k}\left(n\mu Q_S+n\mu Q_M+\frac{\lambda_R\mathcal{N}}{2}I\right)^k[l_0](v)dv\,.$$

We will need to find a uniform bound to $\{C_4(l_k)\}_{k\geq 0}$ for $l_k(v)$ denote $\Lambda^{-k}\left(n\lambda_S Q_S+n\mu Q_M+\mathcal{N}\lambda_R I\right)^k[l_0]$. Lemma 3 below shows that there are constants A' and B' independent of k or n such that $E_{4,k}\leq A'E_{4,0}+B'$ for all k.

We now put everything together. Let $l_0(v)$ be any initial distribution in $L^1(\mathbb{R}^n)$ with finite average 4^{th} moment E_4. Let $l_k(v)$ denote $\Lambda^{-k}\left(n\lambda_S Q_S+n\mu Q_M+\mathcal{N}\lambda_R I\right)^k[l_0]$. By the convexity property of d_2, we have

$$
\begin{aligned}
d_2(e^{-tL_{FR}}[l_0\Gamma_{\mathcal{N}}], e^{-tL_T}[l_0]\Gamma_{\mathcal{N}}) \leq e^{-\Lambda t} \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{k!} \sum_{j=0}^{k-1} \frac{n\mu}{\Lambda} \times& \\
d_2 \left(\left(\frac{\mathcal{N}\lambda_R Q_R + n\lambda_S Q_S + n\mu Q_I}{\Lambda} \right)^j (Q_I l_{k-j-1}\Gamma_{\mathcal{N}}, \right.& \\
\left. \left(\frac{\mathcal{N}\lambda_R Q_R + n\lambda_S Q_S + n\mu Q_I}{\Lambda} \right)^j (Q_M l_{k-j-1}\Gamma_{\mathcal{N}}) \right)& \\
\leq & e^{-\Lambda t} \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{k!} \sum_{j=0}^{k-1} \frac{n\mu}{\Lambda} d_2 \left(Q_I[l_{k-j-1}\Gamma_{\mathcal{N}}], Q_M[l_{k-j-1}]\Gamma_{\mathcal{N}} \right) \\
= & \sup_a e^{-\Lambda t} \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{k!} \sum_{j=0}^{k-1} \frac{\mu}{\mathcal{N}\Lambda} \mathcal{D}_{\mathcal{N}}(H_{\vec{a}}(l_{k-j-1}), a) \\
\leq & \frac{\mu e^{-\Lambda t}}{\mathcal{N}\Lambda} \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{k!} \\
& \times \sum_{j=0}^{k-1} \sup_a [(8C_4(l_{k-j-1}) + \mathcal{D}_1(H_{\vec{a}}(l_{k-j-1}), a)\mathcal{D}_1(H_{\vec{a}}(l_{k-j-1}), a)]^{\frac{1}{2}}
\end{aligned}
$$

We now use the observation that $\mathcal{D}_1(H, a) \leq \sup_{a \neq 0} \mathcal{D}_1(H, a) = d_2(H, 0)$, which is less than or equal to $n\, d_2(Q^I_{i,1} l_{k-j-1}\Gamma_1, M_i l_{k-j-1}\Gamma_1)$. This quantity is clearly not more than

$$
n\left(d_2(Q^I_{i,1} l_{k-j-1}\Gamma_1, \Gamma_{n+1}) + d_2(\Gamma_{n+1}, M_i l_{k-j-1}\Gamma_1)\right) \leq 2n\, d_2(l_0, \Gamma_n).
$$

It still remains to find a uniform upper bound on $\{C_4(l_k)\}$. Equation (4.40) reduces this task to finding a uniform bound on $\{E_{l_{k,4}}\}_{k \geq 0}$. This is done in Lemma 3 below: there are constants A', B' independent of k, n and l_0 such that $E_{l_{k,4}} \leq A' E_{l_{0,4}} + B'$ for all k. Hence we obtain

$$
\begin{aligned}
d_2(e^{-tL_{FR}}[l_0\Gamma_{\mathcal{N}}], e^{-tL_T}[l_0]\Gamma_{\mathcal{N}}) \leq & e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{\Lambda^k t^k}{k!}\sum_{j=0}^{k-1}\frac{\mu}{\mathcal{N}\Lambda}[(8C_4+2n\,d_2(l_{k-j-1},\Gamma_n))2n\,d_2(l_{k-j-1},\Gamma_n)]^{\frac{1}{2}} \\
\leq & e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{\Lambda^k t^k}{k!}\sum_{j=0}^{k-1}\frac{\mu}{\mathcal{N}\Lambda}(1-\frac{\mu}{2\Lambda})^{\frac{(k-j-1)}{2}}[(8C_4+2n\,d_2(l_0,\Gamma_n))2nd_2(l_0,\Gamma_n)]^{\frac{1}{2}} \\
\leq & \frac{1}{\mathcal{N}}\left[e^{-\Lambda t}\sum_{k=1}^{\infty}\frac{\Lambda^k t^k}{k!}\frac{\mu}{\Lambda}\sum_{j=0}^{k-1}(1-\frac{\mu}{4\Lambda})^j\right][(8C_4+2n\,d_2(l_0,\Gamma_n))2nd_2(l_0,\Gamma_n)]^{\frac{1}{2}} \\
= & \frac{1}{\mathcal{N}}[1-e^{-\frac{\mu}{4}t}][(8C_4+2n\,d_2(l_0,\Gamma_n))2nd_2(l_0,\Gamma_n)]^{\frac{1}{2}} \\
= & \frac{2n}{\mathcal{N}}\left(1-e^{-\frac{\mu}{4}t}\right)[(\frac{4C_4}{n}+\,d_2(l_0,\Gamma_n))d_2(l_0,\Gamma_n)]^{\frac{1}{2}} \\
= & \frac{2n}{\mathcal{N}}\left(1-e^{-\frac{\mu}{4}t}\right)\times\left[\left(4(2\pi)^4\left[\max\left\{\frac{3}{8}\left(A'E_{l_0}+B'+\frac{1}{\pi}\right),1\right\}+1\right]\right.\right. \\
& \left.\left.+\,d_2(l_0,\Gamma_n)\right)d_2(l_0,\Gamma_n)\right]^{\frac{1}{2}}.
\end{aligned}
$$

□

We conclude this section by giving the missing lemma. It is interesting that this lemma does not show whether $\Lambda I - L_T$ contracts the fourth moment, or replaces it by the maximum of the fourth moment of our distribution and the fourth moment of the thermostat particle's distribution. Its proof is for any $\beta > 0$.

Lemma 3 *Given a symmetric distribution ϕ_0 on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} v_1^4\phi_0(v)dv = E_4 < \infty$$

we have

$$E_{4,k} = \int v_1^4\phi_k(v)dv \leq A'E_4 + B',$$

where $\phi_k = L^k\phi_0 = \Lambda^{-k}\left(n\lambda_S Q_S + n\mu Q_M + \lambda_R\mathcal{N}I\right)^k\phi_0$ and A' and B' are suitable constants that depend only on E_4 and β. When $\beta = 2\pi$, we can take (A', B') to be $(\sqrt{5}, 1+\frac{3\sqrt{2}}{4\pi^2})$.

Proof.- We will resort to the Hermite polynomials with weight $\Gamma_{1,\beta}(v)$. Recall that

$$\begin{pmatrix} H_0(v) \\ H_2(v) \\ H_4(v) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{\beta} & 1 & 0 \\ \frac{3}{\beta^2} & -\frac{6}{\beta} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ v^2 \\ v^4 \end{pmatrix}, \text{ thus } \begin{pmatrix} 1 \\ v^2 \\ v^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\beta} & 1 & 0 \\ \frac{3}{\beta^2} & \frac{6}{\beta} & 1 \end{pmatrix}\begin{pmatrix} H_0(v) \\ H_2(v) \\ H_4(v) \end{pmatrix}.$$

A suitable basis for the symmetric polynomials of degree ≤ 4 is:

$$\psi_4(v) = \frac{1}{n}\sum_{i=1}^{n} H_4(v_i), \quad \psi_3(v) = \binom{n}{2}^{-1} \sum_{i<j} H_2(v_i)H_2(v_j), \quad \psi_2 = \sum_{i=1}^{n} H_2(v_i), \text{ and } \psi_0(v) = 1.$$

By the symmetry of ϕ_0, and thus of ϕ_k for every k, we have:

$$E_{4,k} = \frac{1}{n}\sum_{i=1}^{n} \int v_i^4 \phi_k(v)\, dv.$$

$$\text{Let} \quad L^k(\frac{1}{n}\sum_{i=1}^{n} v_i^4) = a_{4,k}\psi_4(v) + a_{3,k}\psi_3(v) + a_{2,k}\psi_2(v) + a_{0,k}\psi_0(v).$$

Then $\vec{a}_0 = (a_{4,0}, a_{3,0}, a_{2,0}, a_{0,0})^t$ equals $(1, 0, \frac{6}{\beta}, \frac{3}{\beta^2})^t$, and $\vec{a}_k = M^k \vec{a}_0$ for some matrix M.

Since L is self-adjoint and has norm 1 in $L^2(\Gamma_{n,\beta})$, Lemma 2.7 in [2] says that M also has norm 1. This is because the matrix for M in the $(1, \frac{1}{n}\sum v_i^2, \binom{n}{2}^{-1}\sum v_i^2 v_j^2, \frac{1}{n}\sum v_i^4)$ basis and the matrix for M in the $(\phi_0, \phi_2, \phi_3, \phi_4)$ basis are equal.

Thus, $|\vec{a}_k|_2 \leq |\vec{a}_0| = \sqrt{1 + \frac{36}{\beta^2} + \frac{9}{\beta^4}}$ This is $\leq \sqrt{2}$ when 2π. Next, we have:

$$\begin{aligned}
E_{4,k} &= a_{4,k}\int \psi_4(v)\phi_0(v)dv + a_{3,k}\int \psi_3(v)\phi_0(v)\,dv + a_{2,k}\int \psi_2(v)\phi_0(v)\,dv + a_{0,k} \\
&= a_{4,k}(E_4 - \frac{6}{\beta}E_2 + \frac{3}{\beta^2}) + a_{3,k}\left(E_{2,2} - \frac{2}{\beta}E_2 + \frac{1}{\beta^2}\right) + a_{2,k}\left(E_2 - \frac{1}{\beta}\right) + a_{0,k} \\
&\leq |\vec{a_k}|_2 \left((E_4 - \frac{6}{\beta}E_2 + \frac{3}{\beta^2})^2 + (E_{2,2} - \frac{2}{\beta}E_2 + \frac{1}{\beta^2})^2 + (E_2 - \frac{1}{\beta}))^2 + 1\right)^{\frac{1}{2}}
\end{aligned}$$

with $E_2 = \int \phi_0(v)v^2\,dv$ and $E_{2,2} = \int \phi(v)v_1^2 v_2^2\,dv$. Notice that $E_{2,2} \leq E_4$.

Our claim follows from the following inequalities:

$$\begin{aligned}
(E_4 - \frac{6}{\beta}E_2 + \frac{3}{\beta^2})^2 &\leq (E_4 + \frac{3}{\beta^2})^2 \\
(E_{2,2} - \frac{2}{\beta}E_2 + \frac{1}{\beta^2})^2 &\leq (E_4 + \frac{1}{\beta^2})^2 \\
\text{and} \quad (E_2 - \frac{1}{\beta})^2 &\leq E_4 + \frac{1}{\beta^2} \leq \frac{1}{2}(E_4 + \frac{1}{\beta^2})^2 + \frac{1}{2}.
\end{aligned}$$

Thus, we obtain the bound $E_{4,k} \leq |\vec{a}|_2\sqrt{2.5(E_4 + \frac{3}{\beta^2})^2 + 0.5}$ which, for $\beta = 2\pi$, gives $E_{4,k} \leq \sqrt{2}(\sqrt{2.5}(E_4 + \frac{3}{4\pi^2} + \sqrt{0.5})$. $\square$

Chapter 5

Convergence to Equilibrium Under the GTW metric d_2

The aim of this chapter is to study the relation between the Gabetta-Toscani-Wennberg metric d_2 and the Kac evolution. It is taken from parts of paper [26]. We take our initial distribution μ to be a Borel probability measure on $\mathbb{R}^n$. A special case is a Borel measure supported on $S^{n-1}(\sqrt{nE})$. Equation (1.7) is adapted to measures and we study the Kac-evolved $e^{-tL}\mu$ using the GTW distance d_2.

Section 5.1 gives Theorem 8, a convergence result that provides an upper bound for $d_2(e^{-tL}\mu, R_\mu)$ when μ has zero mean and finite second moment, and is symmetric under the exchange of its variables. This upper bound has the form $\min\{Be^{-\frac{4\lambda_1}{n+3}t}, d_2(\mu, R_\mu)\}$ with B depending only on the second moment of μ as in Proposition 7. This shows that $d_2(e^{-tL}\mu, R_\mu)$ goes to zero. It is curious that the proof uses the gap in (1.13) of the Kac evolution in an unexpected manner. At $t = 0$, Proposition 7 shows that our bound has the correct order of magnitude.

Section 5.2 constructs in Theorem 9 a family of functions $f_n \in L^1(\mathbb{R}^n)$ having $O(t^{n-1})$ decay in d_2 when $0 \leq t \leq 1/(2\lambda)$, using the particular l^∞ nature of the d_2 metric.

5.1 Upper bound for $d_2(e^{-tL}\mu, R_\mu)$

Theorem 8 *Let μ be a Borel probability measure on $\mathbb{R}^n$ that is invariant under permutation of coordinates. Let $\int |v|^2\mu(dv) < \infty$ and $\int v\mu(dv) = \vec{0}$. And let λ in* (1.7) *be* 1. *Then*

$$d_2(e^{-tL}\mu, R_\mu) \leq \min\left\{K\left(e^{-\frac{4\lambda_1}{n-1}t}\right)\left[2\int v_1^2|\mu|(dv) + (n-1)e^{-\frac{4n-6}{n-1}t}\left|\int_{\mathbb{R}^n} v_1v_2\mu(dv)\right|\right], d_2(\mu, R_\mu)\right\}. \tag{5.1}$$

$K = 6.64(2\pi)^2$ *and λ_1 is the gap in* (1.13).

This theorem implies that $d_2(e^{-tL}\mu, R_\mu) \leq 2K(ne^{-2t}+1)\int \frac{|v|^2}{n}\,\mu(dv)\left(e^{-\frac{4\lambda_1}{n+3}t}\right)$ for all t, and that if μ has zero correlations between the v_i (e.g. $\mu = \prod_i \mu_0(dv_i)$ is chaotic with μ_0 centered at 0), then $(ne^{-t}+1)$ can be replaced by 1. The important information in this theorem is the exponential rate of decay $\frac{4\lambda_1}{n+3}$ for large time. The constant K is not optimal at $t=0$. It would be desirable to have a bound of the form $d_2(e^{-tL}\mu, R_\mu) \leq 1e^{-ct/n}d_2(\mu, R_\mu)$. But Theorem 9 implies that no such bound exists on $[0,1/2]$ even if μ has a Schwartz density with respect to the Lebesgue measure.

Proof.- Let μ be a probability measure with mean zero and finite second moment, and let $-L = n(Q-I)$. We use the fact that the Fourier transform commutes with the Kac evolution to take the problem into the Fourier space. Since μ has bounded finite second moment, $\hat{\mu}$ has bounded second derivatives, which allows us to control $|\hat{\mu}-R_{\hat{\mu}}|_{L^\infty(r)}$ by $|\hat{\mu}-R_{\hat{\mu}}|_{L^2(r)}^{\frac{2}{n}}$ on each sphere. The L^2 gap of the Kac operator in [4] gives an exponential decay in $L^2(r)$ for each r, which reflects in a decay in $d_2(e^{-tL}\mu, R_\mu)$ after carefully showing that $\left|e^{-tL}\hat{\mu}-\hat{R}_\mu\right|_{L^\infty(r)}$ is of order r^2 near $r=0$, and after combining the decay results on each sphere.

Let $u(t,\xi)$ be $\hat{\mu}(t,\xi)-\hat{R}_\mu(\xi)$. Proceeding as in equation (3.8) we obtain

$$
\begin{aligned}
\left|\sum_{i,j}\eta_i\eta_j\partial_i\partial_j u(t,\xi)\right| &= \left|-(2\pi)^2\int(\vec{\eta}.v)^2e^{-2\pi i v.\xi}e^{-tL}(\mu-R_\mu)\right| \\
&= (2\pi)^2\left|\int_{\mathbb{R}^n}(\vec{\eta}.v)^2e^{-2\pi iv.\xi}e^{-tL}\left\{(I-R)[\mu]\right\}(dv)\right| \\
&= (2\pi)^2\left|\int_{\mathbb{R}^n}(\vec{\eta}.v)^2e^{-2\pi iv.\xi}(I-R)[e^{-tL}(\mu)](dv)\right| \\
&= (2\pi)^2\left|\int_{\mathbb{R}^n}(I-R)\left\{(\vec{\eta}.v)^2e^{-2\pi iv.\xi}\right\}[e^{-tL}(\mu)](dv)\right| \\
&\leq (2\pi)^2\int(I+R)(\vec{\eta}.v)^2e^{-tL}[\mu](dv) = (2\pi)^2\int e^{-tL}(\vec{\eta}.v)^2\mu(dv)+(2\pi)^2\frac{|v|^2|\eta|^2}{n}\mu(dv) \\
&= (2\pi)^2|\vec{\eta}|^2\left\{2\int v_1^2\mu(dv)+(n-1)e^{-\frac{4n-6}{n-1}t}\left|\int v_1v_2\mu(dv)\right|\right\} \\
&=: L_p(t)|\vec{\eta}|^2 \qquad (5.2)
\end{aligned}
$$

for all ξ, η and all $t \geq 0$. Here we used the commutativity of the radial averaging R with the Kac evolution e^{-tL}, and the fact that both these operators are self-adjoint.

Fix t and $r > 0$. Let $S = S^{n-1}(r)$ and choose $\xi_0 \in S$ and θ_0 so that $e^{-i\theta_0}u(\xi_0) =$

$|u|_{L^\infty(r)}$. All of u, ξ_0, and B (introduced below) depend on t but we suppress this dependence for brevity. First we must show that $|u|_{L^\infty(r)}$ is of order r^2 as $r \to 0$, for d_2 to be bounded. This is accomplished in (5.5) which shows that $|u(\xi_0)| - |u(\xi)|$ is actually quadratic in $|\xi - \xi_0|$ for $\xi \in S$. (5.5) is possible because $|u(\xi)|^2$ has a maximum on S at ξ_0, and thus either $u(\xi_0) = 0$ or $\nabla|u(\xi_0)|$ is perpendicular to S at ξ_0, and thus $\nabla u(\xi_0)$ is parallel to ξ_0. Without loss of generality we can take $u(\xi_0) \neq 0$ for otherwise $u \equiv 0$ on S and S won't contribute to d_2. Let

$$B = S \cap \left\{ |\xi - \xi_0| \leq \sqrt{\frac{|u(\xi_0)|}{3Lp(t)}} \right\}.$$

We show that $|u(\xi)| \geq |u(\xi_0)|/2$ on B. Let η be any point in $\mathbb{R}^n$. By Taylor's theorem we have:

$$u(\eta) = u(\xi_0) + (\nabla u)(\xi_0).(\eta - \xi_0) + \frac{1}{2}\sum_{i,j} \partial_i\partial_j u(\xi^*)(\eta - \xi_0)_i(\eta - \xi_0)_j. \tag{5.3}$$

(5.2) bounds the term $\frac{1}{2}\sum_{i,j} \partial_i\partial_j u(\xi^*)(\eta - \eta_0)_i(\eta - \eta_0)_j$ in absolute value by $\frac{1}{2}Lp(t)|\eta - \xi_0|^2$. To control the linear term in (5.3) on B, we dig into the direction of $\nabla u(\xi_0)$ and show next that under our assumptions, including the assumption that $u(\xi_0) \neq 0$, we have

$$|\nabla u(\xi_0)| \leq Lp(t)|\xi_0|, \tag{5.4}$$

which might be false at other points on S.

(5.4) follows from the fact that at ξ_0, $\nabla u(\xi_0)$ is parallel to ξ_0 since $|u|$ has a maximum there on the sphere. Thus $|\xi_0.\nabla u(\xi_0)| = |\xi_0||\nabla(\xi_0)|$, and at the same time,

$$|\xi_0.\nabla u(\xi_0)| = \left|\sum (\xi_0)_i \int_0^1 \partial_s(\partial_i u)(s\xi_0)\,ds\right| = \left|\sum_i \sum_j (\xi_0)_i(\xi_0)_j \int_0^1 \partial_j\partial_i u(s\xi_0)\,ds\right| \leq |\xi_0|^2 Lp(t)$$

by (5.2), proving equation (5.4).

We can now compute an upper bound for $\nabla u(\xi_0).(\xi - \xi_0)$ on S. Choose a coordinate system in which $\xi_0 = (0, \ldots, 0, 0, r)$ and $\xi = (0, \ldots, w, \sqrt{r^2 - w^2})$. Here we're tacitly assuming that $\vec{\xi}.\vec{\xi_0} > 0$. Equation (5.6) will show that this condition is satisfied on B. Set $e_n = \xi_0/r$. Then $|(\xi - \xi_0).e_n| = |r - \sqrt{r^2 - w^2}| = \frac{w^2}{r + \sqrt{r^2 - w^2}} \leq \frac{w^2}{r}$. Similarly, $|\xi - \xi_0|^2 = w^2 + (r - \sqrt{r^2 - w^2})^2 = 2r^2(1 - \sqrt{1 - \frac{w^2}{r^2}}) \geq w^2$, which, together with equation (5.4) gives us:

$$|(\nabla u)(\xi_0).(\xi-\xi_0)| \leq Lp(t)r \times \frac{w^2}{r} \leq Lp(t)|\xi-\xi_0|^2.$$

To summarize, we have shown that

$$|u(\xi_0)-u(\xi)| \leq \frac{3}{2}Lp(t)|\xi-\xi_0|^2, \quad \text{for all } \xi \in S \cap \{\eta.\xi_0 \geq 0\}. \tag{5.5}$$

This implies that $|u(\xi)| \geq |u(\xi_0)| - \frac{3}{2}Lp(t)|\xi-\xi_0|^2 \geq \frac{|u(\xi_0)|}{2}$ on $B \cap \{\eta.\xi_0 \geq 0\}$.

$\xi.\xi_0$ is positive on B because of equation (5.6):

$$|u(\eta,t)| \leq \frac{Lp(t)}{2}|\eta|^2 \qquad \text{for any } \eta \text{ and } t, \tag{5.6}$$

which follows from (5.3) at $\xi_0 = 0$ and the fact that $u(t,0) = \int \mu(dv) - \int R_\mu(dv) = 0$ and $\nabla u(t,0) = -2\pi i \int ve^{-tL}\mu(dv) - \vec{0} = \int e^{-Lt}v\mu(dv) = \vec{0}$.

A simple computation is left to prove (5.1). Choose a coordinate system in which ξ_0 points towards the North Pole and denote by θ the angle from ξ_0. We see that the largest value of θ for points in B is given by the equation

$$|\xi-\xi_0|\text{max} = 2r\sin(\frac{1}{2}\theta\text{max}).$$

Integrating out the rest of the angular variables in σ^r we obtain

$$\begin{aligned}
\sigma_r(B) &= \frac{\int_0^{2\sin^{-1}\left(\sqrt{\frac{|u(\xi_0)|}{12r^2\,Lp(t)}}\right)} \sin(\theta)^{n-2}\,d\theta}{\int_0^\pi \sin(\theta)^{n-2}\,d\theta} \geq \frac{\int \sin(\theta)^{n-2}\cos(\theta)\,d\theta}{\int_0^\pi \sin(\theta)^{n-2}\,d\theta} \\
&= \frac{\left(4\frac{|u(\xi_0)|}{12r^2Lp(t)}\left(1-\frac{|u(\xi_0)|}{12r^2Lp(t)}\right)\right)^{(n-1)/2}}{(n-1)\int_0^\pi \sin(\theta)^{n-2}\,d\theta} \geq \frac{\left(\frac{23}{72}\frac{|u(\xi_0)|}{Lp(t)r^2}\right)^{(n-1)/2}}{(n-1)\int_0^\pi \sin(\theta)^{n-2}\,d\theta}.
\end{aligned}$$

This gives us the lower bound $||u(t,\xi)||^2_{L^2(r)} \geq \dfrac{|u(t,\xi_0)|^2}{4}\sigma_r(B)$. Letting $b(t,r) := \frac{|u(t,\xi_0)|}{r^2Lp(t)}$, we obtain: $b \leq \frac{1}{2}$ for all t thus

$$\frac{||u(t,.)||^2_{L^2(r)}}{(Lp(t)r^2)^2} \leq e^{-2\lambda_1 t}\frac{||u(0,.)||^2_{L^2(r)}}{(Lp(t)r^2)^2} \leq e^{-2\lambda_1 t}\frac{||u(0,.)||^2_{L^\infty(r)}}{(Lp(t)r^2)^2} \leq \left(\frac{Lp(0)}{Lp(t)}\right)^2\frac{e^{-2\lambda_1 t}}{4}. \tag{5.7}$$

at the same time we have

$$\frac{||u(t,.)||^2_{L^2(r)}}{(Lp(t)r^2)^2} \geq \frac{|u(t,\xi_0)|^2}{4(Lp(t)r^2)^2}\sigma_r(B) \geq \frac{b(t,r)^2}{4}\frac{\left(\frac{23}{72}b(t,r)\right)^{(n-1)/2}}{(n-1)\int_0^\pi \sin(\theta)^{n-2}\,d\theta} \tag{5.8}$$

Hence

$$b(t,r) \leq \frac{72}{23}e^{-\frac{4\lambda_1}{n+3}t}\left((n-1)\left(\frac{n+1}{ne^{-t}+1}\right)^2\int_0^\pi \sin(\theta)^{n-2}\,d\theta \times \left(\frac{72}{23}\right)^{\frac{n-1}{2}}\right)^{2/(n+3)}.$$

Finally, since

$$\left(\frac{23^2}{72^2}\frac{(k-1)(k+1)^2}{(ke^{-t}+1)^2}\int_0^\pi \sin\theta^{k-2}\,d\theta\right)^{\frac{2}{k+3}}$$

is maximized on $\{k \geq 2, t \geq 0\}$ when $k = 6$ and $t = \infty$, it is less than 2.1207 and we have $b(t,r) \leq \dfrac{72}{23} \times 2.1207e^{-\frac{4\lambda_1}{n+3}}$ and $d_2(e^{-tL}\mu, R_\mu) \leq 6.64Lp(t)e^{-\frac{4\lambda_1}{n+3}t}$. □

Remark 4 *The proof of Theorem 8 relies on* (5.7) *and* (5.8) *which can be seen as L^∞ being interpolated between $(L^2)^{\frac{1}{n}}$ and $W^{2,\infty}$. It is good that we used $Lp(t)$ in our bounds instead of $Lp(0)$. This potentially saves a factor n for some initial distributions. It would be interesting function-analytically to see if the exact form of u can bring in more information to this interpolation inequality.*

5.2 Slow initial Decay in d_2

Theorem 9 *Let $n \geq 2$ and let L be as in* (1.7) *with $\lambda = 1$. Then there is a Schwartz probability density f_n on $\mathbb{R}^n$ that satisfies*

$$d_2(e^{-tL}f_0, R_{f_0}) \geq \max\left\{d_2(f_0, R_{f_0})\left(1 - \frac{e}{n}(2t)^{n-1}\right), 0\right\} \text{ for all } t \geq 0. \tag{5.9}$$

The lower bound in Theorem 9 endures for $t \in [0, 1/2]$. The f_n are given explicitly in Lemma 6 up to two parameters $\bar{A}(n)$ and $B(n)$ that are shown to be finite but are not computed. The functions f_n will be perturbations of the Gaussians $\prod_{i=1}^n \Gamma_{\alpha(n)}(v_i)$ at high temperature by Schwartz functions that have small L^1 norms.

The Theorem says that no matter how large n is, $d_2(e^{-tL}f_n, R_{f_n})$ is practically unchanged for time $= \frac{1}{2\lambda}$. Although this result provides no information about the decay after time of order

1, it does rule out bounds of the form $d_2(f(t), R_f) \leq e^{-ct} d_2(f(0), R_f)$ for any c. Let us rescale the time so that $\lambda = 1$.

Proof.- The proof starts with constructing an auxiliary function ψ in Lemmas $1-3$. These ψ are Schwartz functions with integral zero that satisfy

$$d_2(Q^k\psi, R_\psi) = d_2(\psi, R_\psi) \text{ for } k = 0, 1, \ldots, n-2. \tag{5.10}$$

Then these ψ are scaled. After this a positive Gaussian at large enough temperature is added to them to obtain the non-negative functions f_n. The existence of ψ satisfying (5.10) is not very surprising. It follows from the L^∞ nature of the d_2 metric and the fact that it takes $n-1$ Kac rotations Q of a vector v to cover the whole sphere $|w| = |v|$. This is analogous to the result in [1] where it is shown that the total variation distance between an initial permutation of a deck of cards and the uniform distribution is not affected by $O(\ln(n))$ riffle-shuffles. The reason for this invariance is the fact that starting from a fixed initial distribution there are permutations of cards which cannot be reached in less than $O(\ln(n))$ riffle-shuffles.

Since d_2 deals with the Fourier transform, the fact that the Fourier transform commutes with rotations, and thus with the Kac rotations $Q_{i,j}$, is a major help. We can directly construct the Fourier transform of the f_n-s and only afterwards worry about the nonnegativity requirement on f_n and the condition that the f_n need to belong to $L^1(\mathbb{R}^n)$. First we construct a one parameter family of functions $\phi(\xi;\alpha) \geq 0$ such that $Q^k\phi((z,0,0,\ldots,0);\alpha) = 0$ for all z, α and all $k \leq n-2$. Then we modify $\phi(\xi)$ slightly to $A(|\xi|)\phi(\xi;\alpha)$ to make sure that $\sup \dfrac{|A(|\xi|)\left\{\phi(\xi;\alpha) - R_\phi\right\}|}{|\xi|^2}$ is finite and occurs away from the origin on $\{|\xi|^2 \geq b\}$ with b to be specified below. Taking $|\xi|^2 \geq b$ one can adjust α so that $|A(r)\left[Q^k\left[\phi(\xi;\alpha)\right] - R_\phi(\xi;\alpha)\right]|_{L^\infty(r)} = A(r)R_\phi(r;\alpha)$ for all $k \leq n-2$. This will be the case when the $L^\infty(r)$ is attained at $(\pm r, 0, 0, \ldots, 0)$.

Let $h(x;\alpha) = (1 - e^{-\alpha x^2})$ and set $\phi(\xi;\alpha) = \prod_{i=1}^n h(\xi;\alpha)$. Then we have:

Lemma 4 *Properties of* ϕ

Fix $|\xi| = r$*, and let* $z_1 = (r, 0, 0, \ldots, 0)$*. Then for all* $l \leq n-2$ *we have*

(a) $[Q^l\phi](z_1) = \phi(z_1) = 0$*,*

(b) $R_\phi(z_1) > \frac{1}{2}|\phi|_{L^\infty(r)}$*; provided* $\alpha \geq \alpha(r)$ *is large enough, and*

(c) $\frac{(nt)^{n-1}}{(n-1)!} Q^{n-1}\phi(z_1) \leq \frac{e}{n}(2t)^{n-1}|\phi|_{L^\infty(r)}$.

(d) $|\phi|_{L^\infty(r)} = (1-e^{-\alpha r^2/n})^n$.

Remark 5 *Properties* (a) *and* (c) *are easier to prove for the function* $\prod_{i=1}^n \xi_i^2$. *We use* $h(x;\alpha)$ *instead of* x^2 *in the definition of* ϕ *to satisfy property* (b). (b) *tells us that*

$$|\phi(\xi) - R_\phi| \text{ is maximized on } S^{n-1}(r) \text{ at } (\pm z_1, 0, \ldots, 0),$$

since

$$R_\phi(r) - |\phi|_{L^\infty(r)} \leq R_\phi(\xi) - \phi(\xi) \leq R_\phi(r)$$

thus, on the sphere $S^{n-1}(r)$ *we have*

$$\begin{aligned} |\phi(\xi) - R_\phi| &\leq \max\left\{R_\phi(r), \left|R_\phi(r) - |\phi|_{L^\infty(r)}\right|\right\} \\ &= R_\phi(r) \end{aligned}$$

because of property (b).

Remark 6 *The weight of* $Q^{n-1}[\phi](z_1)$ *comes from the Taylor expansion of* $e^{-n(I-Q)t}\phi$.

Proof.-

- Part (a)

  Given a sequence of Kac rotations $Q_{i_1,j_1}(\theta_1), \ldots, Q_{i_k,j_k}(\theta_k)$, we can define a sequence of trigonometric polynomials as follows:

$$\begin{pmatrix} P_1^{(0)} \\ P_2^{(0)} \\ \cdots \\ P_n^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}.$$

  Once $\{P_i^{(s)}\}_{i=1}^n$ are defined, define $P_i^{s+1}(\theta_1, \ldots, \theta_k)$ as follows:

$$P_i^{(s+1)} = \begin{cases} P_i^{(s)}(\theta_1, \ldots, \theta_k), & i \notin \{i_{s+1}, j_{s+1}\} \\ P_i^{(s)}(\theta_1, \ldots, \theta_k)\cos(\theta_{s+1}) - P_{j_{s+1}}^{(s)}(\theta_1, \ldots, \theta_k)\sin(\theta_{s+1}), & i = i_{s+1} \\ P_i^{(s)}(\theta_1, \ldots, \theta_k)\sin(\theta_{s+1}) + P_{i_{s+1}}^{(s)}(\theta_1, \ldots, \theta_k)\cos(\theta_{s+1}), & j = j_{s+1} \end{cases}$$

We are interested in these polynomials since they give the velocity of particle 1 after the k Kac collisions above as follows:

$$v_1(\text{ after }) = \sum_{i=1}^{n} P_i^{(k)}(\theta_1, \ldots, \theta_k) v_i(\text{initial}).$$

We now show that if $i \geq 2$ is an index for which the "edges" $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ do not connect "vertex" i to "vertex" 1, then $P_i(\theta_1, \ldots, \theta_k) = 0$. Let $\mathcal{G}$ denote the graph on $(v_1, \ldots, v_n)$ with edges $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. Let C be the connected component of v_i. An easy inductive argument shows that $\{P_j^{(l)} : j \in C\}$ depends only on $\{P_j^{(0)} : j \in C\}$, for $l = 0, 1, \ldots, k$. In particular, $P_i^{(k)}$ is obtained from $\{P_j^{(0)}(\theta_1, \ldots, \theta_k) : j \in C\}$ after possibly multiplying them by $\cos\theta$-s and $\sin\theta$-s, and adding them up. Since $P_j^{(0)} \equiv 0$ for $j \in C$, we have $P_i^{(k)}(\theta_1, \ldots, \theta_k) \equiv 0$. As a conclusion, it follows that if

$$Q_{i_k,j_k} \ldots Q_{i_1,j_1} \prod \left(1 - e^{-\alpha\xi_i^2}\right)\Big|_{z_1} = \frac{1}{(2\pi)^k} \int \prod_{i=1}^{n} \left(1 - e^{-\alpha r^2 (P_i^{(k)}(\{\cos(\theta_l), \sin(\theta_l)\})^2}\right) \prod_{j=1}^{k} d\theta_j :\neq 0$$

then the connected component C of i contains 1 for each i. So $\mathcal{G}$ is a connected graph which means that $k \geq n - 1$. Part (a) follows from the hypothesis that $k \leq n - 2$.

- Part (b)

  For $r > 0$ and $n \geq 2$ fixed,

$$\frac{\phi}{|\phi|_L^\infty(r)} = \frac{\prod(1 - e^{-\alpha\xi_i^2})}{(1 - e^{-\alpha r^2/n})^n} \to 1$$

almost everywhere on $S^{n-1}(r)$ as $\alpha r^2 \to \infty$. Thus, by the dominated convergence theorem, there exists an $\bar{A}(n) < \infty$ such that if $\alpha r^2 \geq \bar{A}(n)$ then $\int_{S^{n-1}(r)} \phi(w)\sigma^r(dw) \geq \frac{1}{2}|\phi|_{L^\infty(r)}$. Let

$$\alpha(r, n) = -\frac{\bar{A}(n)}{r^2}. \tag{5.11}$$

Note that this property of having $L^1(r)$ norm greater than or equal to $\frac{1}{2}$ of the $L^\infty(r)$ norm propagates in time under the Kac evolution e^{-tL}. This is because the Kac evolution does not change the L^1 norm of positive functions, but decreases (or keeps constant) its L^∞ norm. This observation is also true when we replace e^{-tL} by Q^k.

- Part (c)

By Cayley's theorem there are n^{n-2} distinct trees on n vertices, and for each tree we can order its edges in $(n-1)!$ ways. And each order of presentation of the edges in the tree comes with a weight $\binom{n}{2}^{-(n-1)}$. The terms $Q_{i_{n-1},j_{n-1}}\dots Q_{i_1,j_1}[\phi](z_1)$ where the edges $\{(i_1,j_1),\dots,(i_{n-1},j_{n-1})\}$ do not connect all the vertices $(v_1,\dots,v_n)$ evaluate to zero. The rest of the terms are non-negative and bounded above by $|\phi|_{L^\infty(r)}$. Thus,

$$\frac{(nt)^{n-1}}{(n-1)!}(Q^{n-1}\phi)(z_1) \le \frac{(nt)^{n-1}}{(n-1)!}\frac{(n-1)!n^{n-2}}{\binom{n}{2}^{n-1}}|\phi|_{L^\infty(r)} \le \frac{e}{n}(2t)^{n-1}|\phi|_{L^\infty(r)},$$

proving (c).

- Property (d) follows from an application of the method of Lagrange multipliers. $\square$

In the above lemma $\alpha(r,n)$ is proportional to r^{-2}. Thus, we should find a way to keep $r=|\xi|$ strictly away from zero when d_2 is being evaluated. One way to do this is to consider the product $\phi(\xi)A(\xi) =: \psi(\xi)$; where $A(\xi)=|\xi|^4e^{-|\xi|^2}$. The following lemma makes this claim more precise.

Lemma 5 *Let $A(\xi)=|\xi|^4e^{-|\xi|^2}$. Let b be the smaller solution to $(xe^{-x}=\frac{1}{2}e^{-1})$ ($b\approx 0.23196$). And let $\alpha=\alpha(\sqrt{b},n)$ be as in* (5.11). *Take $\psi=A(\xi)\phi(\xi)$. Then $\frac{|\psi-R_\psi|}{|\xi|^2}$ has a maximum at a point of the form $(x,0,0,\dots,0)$, with $x^2\ge b$.*

Proof.- Let α be as in the hypothesis. Then $R_\phi(\xi)\ge\frac{1}{2}$ when $|\xi|\ge\sqrt{b}$ by property (b) of Lemma 4. In particular: $\frac{|\psi(1,0,\dots,0)-R_\psi(1)|}{|1|^2}=e^{-b}\frac{R_\phi(1,0,0,\dots,0)}{1}\ge\frac{1}{2}e^{-b}\ge\frac{1}{2}e^{-1}$. So if $|\xi|^2<b$, then $\frac{|\psi(\xi)-R_\psi(\xi)|}{|\xi|^2}<be^{-b}<\frac{1}{2}e^{-1}$. So we know that $\max\frac{|\psi-R_\psi|}{|\xi|^2}$ is attained at a point $\vec{\xi}$ with norm at least $\sqrt{b}$. But in this case, for our choice of α, we have $R_\phi\ge\frac{1}{2}|\phi|_{L^\infty(r)}$ and Lemma 4:(a) shows that ξ can be taken to have the form $(x,0,\dots,0)$ for some $x\ge\sqrt{b}$. $\square$

We now give an explicit formula for f_0.

Lemma 6 *Let $b,\alpha=\alpha(\sqrt{b},n)$ be as in Lemma 5 and* (5.11). *Set*

$$f_0(v)=\left(\frac{0.9\pi}{1+\alpha}\right)^{\frac{n}{2}}e^{-\left(\frac{0.9\pi^2}{1+\alpha}\right)|v|^2}+\frac{1}{B(2\pi)^4}\triangle^2\prod_{i=1}^{n}\left(\sqrt{\pi}e^{-\pi^2v_i^2}-\sqrt{\frac{\pi}{1+\alpha}}e^{-\frac{\pi^2}{1+\alpha}v_i^2}\right)$$

for B large enough described below. Then f_0 is a probability density and equation (5.9) *holds for f_0.*

Proof.- Notice that $f_0(v)$ is the sum of a Gaussian and $\frac{1}{B}\check{\psi}$. The Gaussian is radial at a high temperature since α is large. For large $|v_i|$, $\check{\psi}$ is bounded by $\exp(-\frac{\pi^2}{1+\alpha}|v|^2)$ times (a polynomial of degree 4). So there exists a constant $B = B(n)$ that makes $|\check{\psi}| \le B\left(\frac{0.9\pi}{1+\alpha}\right)^{\frac{n}{2}} e^{-\left(\frac{0.9\pi^2}{1+\alpha}\right)|v|^2}$. This shows that $f_0 \ge 0$. Since ψ is a Schwartz function, $\int \check{\psi}(v)\, dv = \psi(0) = 0$. This shows that f_0 integrates to 1.

We now prove (5.9) for f_0. Note that $\frac{\left|e^{-tL}\hat{f}_0(\xi) - \hat{R}_{f_0}(\xi)\right|}{|\xi|^2} = \frac{\left|e^{-tL}\psi(\xi) - R_\psi\right|}{B|\xi|^2}$. We showed in Proposition 5 that at $t = 0$, this term is maximized at a point $z_1 = (z_0, 0, 0, \ldots, 0)$ for some $z_0 \ge \sqrt{b}$. Fix $k \le n-2$. Then

$$\begin{aligned} 0 \;&\ge\; \frac{d_2(e^{-tL}f_0, R_{f_0}) - d_2(f_0, R_{f_0})}{t^k} = \frac{d_2(e^{-tL}f_0, R_{f_0}) - \frac{R_\psi(z_1)}{B\,|z_1|^2}}{t^k} \\ &\ge\; \frac{1}{Bt^k}\frac{(R_\psi(z_1) - e^{-tL}\psi)(z_1)) - R_\psi(z_1)}{z_0^2} = -\frac{e^{-tL}\psi(z_1)}{Bt^k z_0^2} = -\frac{z_0^2 e^{-z_0^2}}{Bt^k} e^{-tL}\phi(z_1) \end{aligned}$$

In the last equality we used the fact that $e^{-tL}\psi$ and ψ have the same radial parts.

We showed in Lemma 4 below that $Q^l\phi(z_1) = \phi(z_1) = 0$ for $l = 0, 1, 2, \ldots, n-2$. Hence, the same is true for their linear combinations $n^k(I-Q)^k$. Thus, as $t \to 0^+$, the right-hand side converges to $-\frac{1}{z_0^2}\left(\frac{n^k}{n!}(I-Q)^k(\phi)(z_1)\right)$ which is also zero.

By Taylor's theorem in t, we have $\left(e^{tnQ}\phi\right)(z_1) = \frac{n^{n-1}t^{n-1}}{(n-1)!}e^{t^*nQ}Q^{n-1}(\phi)(z_1)$ for some $t^* \in (0, t)$. Thus:

$$\begin{aligned} 0 \;&\ge\; \frac{d_2(e^{-tL}f, R_f) - d_2(f, R_f)}{t^{n-1}} = \frac{d_2(e^{-tL}f, R_f) - \frac{R_\psi(z_1)}{B\,z_0^2}}{t^{n-1}} \\ &\ge\; -\frac{z_0^2 e^{-z_0^2}}{Bt^{n-1}} e^{-tL}\phi(z_1) = -\frac{z_0^2 e^{-z_0^2}}{B}\frac{n^{n-1}}{(n-1)!}\frac{e^{-nt}e^{t^*nQ}Q^{n-1}\phi(z_1)}{t^{n-1}} \end{aligned}$$

Finally, since $\left(e^{t^*nQ}Q^{n-1}\phi\right)(z_1)$ is less than $e^{t^*n}|Q^{n-1}\phi|_{L^\infty(S(z_0))}$, we conclude that

$$\frac{d_2(e^{-tL}f, R_f) - d_2(f, R_f)}{t^{n-1}} \ge -\frac{n^{n-1}}{(n-1)!} z_0^2 e^{-z_0^2}\frac{|Q^{n-1}\phi|_{L^\infty(z_0)}}{B}.$$

This together with property (c) in Lemma 4 gives (5.9). $\square$

Chapter 6

Conclusion and Outlook

In this thesis, we studied the approach to equilibrium in the Kac model under various metrics and thermostating conditions. We studied in chapter 2 the partially thermostated Kac model with a strong thermostat. The L^2 spectral gap of the generator of the Kac evolution was proved to be of order $\alpha = \frac{m}{n}$, the fraction of the thermostated particles. This gives an exponential convergence to the equilibrium state at the same rate. And, if the fraction of thermostated particles is fixed, this rate survives the $n \to \infty$ limit. We also gave an upper bound for convergence to equilibrium using relative entropy. This upper bound is essentially exponential at a rate $O\left(\frac{m}{n^2}\right)$, which vanishes in the $n \to \infty$ limit when the fraction of particles α is fixed.

We also studied in chapter 4 the validity of the infinite thermostat assumption used in the model in [2], and showed that in the L^2 metric and, under a technical finite 4^{th} moment assumption, in the d_2 metric, that Kac's evolution using a finite reservoir (4.5) remains close to Kac's evolution using the infinite thermostat under the suitably chosen collision rates in equation (4.4).

We saw in Theorem 8 that under the Kac evolution a Borel measure μ approaches its angular average R_μ in the GTW metric d_2 exponentially with rate at least $O\left(\frac{1}{n}\right)$ and showed in Theorem 9 that the initial decay in d_2 can be virtually zero at least for time $1/(2\lambda)$ which is a macroscopic quantity. As a side, we showed that the average energy per particle also controls $d_2(\mu, R_\mu)$ after time of order $\ln(n)$. The method of proof of Theorem 8 gives an application of the L^2 gap to initial states that are not necessarily in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. The proof of Theorem 8 can be generalized to any evolution which has an L^2 gap and which commutes with the Fourier transform. For example: the Kac model in 1 dimension with initial states not symmetric in its variables; it has a gap independent of n as shown in [13] and [4]; and the Kac model in 1 dimension with symmetric collision rules for which θ in (1.5) has weight $\rho(\theta)$ where ρ is not necessarily constant but satisfies the local reversibility condition $\rho(2\pi - \theta) = \rho(\theta)$.

A lot of questions in the field remain open. Can one prove an exponential decay rate for the partially thermostated Kac model that depends only on α for an extensive distance like $S(.|\Gamma_{n,\beta})$ or an intensive metric like $d_2(., \Gamma_{n,\beta})$ at least for a class of physically interesting initial states? For the special case of $\alpha = 1$, this follows immediately from results for the weak thermostat in [2].

The strong thermostat P was used to simplify the computations in the proofs of Theorems 2 and 3. But all that is proven with the strong thermostat should be true for the Maxwellian thermostat M as suggested by the van-Hove relation in 4 and the relation (2.18). This is still open, but within reach.

An interesting aspect of the proof of the entropy decay in [25] for the model in(1.18) is that we worked directly with entropy and used Jensen's inequality. The bound (2.17) for $S(h(t))$ has zero derivative at $t = 0$. Even for the case $n = 2$ and $m = 1$, it is not known whether there is a strictly positive entropy production at $t = 0$. The regular convexity argument alone is not enough as suggested in Appendix A2.

The sequence of functions $\{f_n\}$ inspires a set of questions. Can there be a sequence of distributions μ_n similar to $\{f_n\}$ except that the μ_n are supported on the spheres $S^{n-1}(\sqrt{nE})$? Also, the f_n are small-L^1 and Schwartz perturbations of Gaussians at high temperature. These perturbations have a very particular algebraic structure. It is not known if there is a physical interpretation to these perturbations, or if there are other perturbations for which there is a nice physical interpretation.

The constant K_μ in Theorem 8 is not optimal as Proposition 7 suggests. This raises the question of what the optimal $K_\mu(n)$ is. Proposition 7 suggests that for a measure that is symmetric in its variables, $d_2(\mu, R_\mu) \leq H(\mu, n)$ for some optimal $H(\mu, n)$ which is at most linear in $\int v_1^2 \mu(dv)$ and in $\left|\int v_1 v_2 \mu(dv)\right|$. The following candidate

$$K_\mu(n) = \left(1 + \frac{d}{n}\right) H(\mu, n) \tag{6.1}$$

is consistent with Theorem 9 since $(1 + \frac{d}{n})e^{-\frac{t}{n}}$ shows decay after time of order 1, as in Theorem 9 instead of time of order n required when 1 is replaced by $1 + \epsilon$ with ϵ independent of n. It is open whether (6.1) is true for a large class of Borel measures μ.

Finally, some general questions. Will any initial state take time of order n to decay significantly? The L^2 gap (2.9), the entropy production of order $\frac{1}{n}$ in [21] and [10], and Theorem 8 do not rule this out. Equation (2.9) only says that the decay in L^1 becomes $O(e^{-\lambda_1 t})$ only after some time, which could possibly be of order $n\ln(n)$. Another interesting task is to study the convergence to equilibrium in the momentum preserving Kac model in 3D, still with Maxwellian molecules, using d_2, and for Kac collisions of non-Maxwellian molecules, where the collision $Q_{i,j}$ takes place at a rate proportional to $(v_i^2+v_j^2)^{\frac{\gamma}{2}}$ for some $\gamma \in (0,2]$. Ideally, one would like to study a spatially inhomogeneous model with possibly non-Maxwellian moleculesl where collision rate between particles i and j is proportional to $|v_i - v_j|^\gamma$.

Appendix A1

This appendix gives the spectrum of $\bar{L}_{2,1} = 2\lambda(I-Q) + \mu(I-\bar{P}_1)$ in $L^2(\mathbb{R}^2, \Gamma_{2,\beta}(v)\,dv)$ and, in particular, finds $\Delta_{2,1}$ as stated in (2.9). For simplicity, we denote the operators $L_{2,1}$ and P_1 by L and P.

Notice that $\bar{L}$ is a linear combination of two projections ($Q \equiv Q_{12}$ is an orthogonal projection onto radial functions in $\mathbb{R}^2$). The condition $\langle h, 1\rangle = 0$ corresponding to the normalization of $f = \Gamma(1+h)$, and leads us to work in the space of Hermite polynomials $\{H_\alpha(v)\}_{\alpha=0}^\infty$ with weight $\Gamma_1(v)$. The space of interest $\{h \in L^2(\mathbb{R}^2, \Gamma_2) : \int h\Gamma_2(v)\,dv = 0\}$ is spanned by $\{K_{i,j} : i,j \in \mathbb{N}, (i,j) \neq (0,0)\}$, where $K_{i,j} := H_i(v_1)H_j(v_2)$. With H_i the monic Hermite polynomials.

The action of $\bar{P}$ is as follows:

$$\bar{P}K_{i,j} = \begin{cases} 0 & : i \neq 0 \\ K_{0,j} & : i = 0 \end{cases}$$

Since each term in $K_{i,j}$ is odd in either v_1 or v_2 when either i or j is odd, we have that $QK_{i,j} = 0$ when either i or j is odd. We deduce the action of Q on $K_{2\alpha_1,2\alpha_2}$ from its action on $v_1^{2\alpha_1}v_2^{2\alpha_2}$ using the following lemma from [2], which applies to Q as it is a projection onto radial functions.

Lemma 7 *[2] Let A be a self-adjoint operator on $L^2(\mathbb{R}^N, \Gamma(v)dv)$ that preserves the space P_{2l} of homogeneous even polynomials in $v_1, ..., v_N$ of degree $2l$. If*

$$A(v_1^{2\alpha_1}...v_N^{2\alpha_N}) = \sum_{\sum \alpha_i = \sum \beta_i} c_{\beta_1...\beta_N} v_1^{2\beta_1}...v_N^{2\beta_N}\ ,$$

we get

$$A(H_{2\alpha_1}(v_1)...H_{2\alpha_N}(v_N)) = \sum_{\sum \alpha_i = \sum \beta_i} c_{\beta_1...\beta_N} H_{2\beta_1}(v_1)...H_{2\beta_N}(v_N)\ .$$

Let $d := \alpha_1 + \alpha_2$ and $b_{2\alpha_1,2\alpha_2} := \fint_0^{2\pi} \cos^{2\alpha_1}\theta \sin^{2\alpha_2}\theta d\theta = \frac{(2\alpha_1-1)!!(2\alpha_2-1)!!}{2^{\alpha_1+\alpha_2}(\alpha_1+\alpha_2)!}$, with the standard definition $(-1)!! = 1$. Then we have

$$QK_{i,j} = \begin{cases} 0 & : i \text{ or } j \text{ odd} \\ b_{2\alpha_1,2\alpha_2} \sum_{l=0}^{d} \binom{d}{l} K_{2l,2d-2l} & : i = 2\alpha_1, j = 2\alpha_2 \end{cases}$$

By Lemma 7 This last term equals $(v_1^2 + v_2^2)^d$, the radial polynomial in 2 variables of degree d. Using the fact that $L_{2d} := \mathrm{Span}\{H_{2\alpha_1}(v_1)H_{2\alpha_2}(v_2) : \alpha_1 + \alpha_2 = d\}$ are invariant subspaces for L, yields the following for the spectrum of L:

Eigenvalue	Eigenfunction
$2\lambda + \mu$	$K_{i,j}$, i or j odd, $i \neq 0$ $\sum_{i=1}^{d} c_i K_{2i,2d-2i}$ where $\sum_{i=1}^{d} c_i b_{2i,2d-2i} = 0$
2λ	$K_{0,j}$, j odd
$x^{\pm,d}$	$\sum_{i=0}^{d} c_i^{\pm,d} K_{2i,2d-2i}$ and eq. (6.2)

Remark 7 *The first row corresponds to functions that belong to the kernels of both Q and P, and the second row to functions that belong to the kernels of Q and $I - P$.*

Here,

$$x^{\pm,d} = \frac{(2\lambda + \mu) \pm \sqrt{(2\lambda + \mu)^2 - 8\lambda\mu(1 - b_{0,d})}}{2}$$

and

$$c_0^{\pm,d} = \frac{2\lambda}{2\lambda - x^{\pm,d}} \text{ and } c_i^{\pm,d} = \frac{2\lambda\binom{d}{i}}{x^{\mp,d}} \text{ for } i \neq 0 \tag{6.2}$$

Using the fact that $b_{0,2d} = \fint_0^{2\pi} \cos^{2d}\theta d\theta$ is decreasing in d, it is easy to see that the smallest eigenvalue is $x^{-,1}$. The corresponding eigenfunction is $\frac{2\lambda}{2\lambda - x^{-,1}} K_{0,2} + \frac{2\lambda}{x^{+,1}} K_{2,0}$.

Proof.- The case when i or j is odd is clear. So we consider $h = \sum_{i=0}^{d} c_i K_{2i,2d-2i}$. Let h be an eigenfunction of $2\lambda Q + \mu \bar{P}$ with eigenvalue x. Then

$$2\lambda(\sum_{i=0}^{d} c_i b_{2i,2d-2i}) \sum_{i=0}^{d} \binom{d}{i} K_{2i,2d-2i} + \mu c_0 K_{0,2d} = x \sum_{i=0}^{d} c_i K_{2i,2d-2i}.$$

For simplicity, let $K = \sum_{i=0}^{d} c_i b_{2i,2d-2i}$. We have

$$\begin{cases} 2\lambda K\binom{d}{i} = xc_i, & i \neq 0 \\ 2\lambda K + \mu c_0 = xc_0 \end{cases}.$$

There are two cases:

1. $K = 0$. Then, Either $c_i = 0$ for all $i \neq 0$. Thus $x = \mu$ and the eigenvalue is $(2\lambda+\mu)-x = 2\lambda$. Or, $x = 0$ and thus $c_0 = 0$ and the eigenvalue is $2\lambda + \mu$. In this case, $\sum_{i=1}^{n} c_i b_{2i,2d-2i}$ has to be zero. This gives the first two rows of the table.

2. $K \neq 0$. By the invariance of an eigenfunction under scaling, we can choose $K = 1$. Then

$$c_i = \frac{2\lambda\binom{d}{i}}{x} \; i \neq 0, \text{ and } c_0 = \frac{2\lambda}{x-\mu}$$

   Therefore

$$1 = K = \sum_i c_i b_{2i,2d-2i} = \frac{2\lambda}{x}\sum_{i=1}^{d} b_{2i,0}\binom{d}{i} + \frac{2\lambda}{x-\mu} = \frac{2\lambda}{x}(1-b_{2d,0}) + \frac{2\lambda b_{2d,0}}{x-\mu}$$

This gives the missing eigenvalues and the gap in (2.9) □

Appendix A2

For every ϵ in $(0,1]$ there is a probability density $f_\epsilon = h_\epsilon \Gamma_{2,\beta}(v_1, v_2)$ with $h \in L^2(\mathbb{R}^2, \Gamma_{2,\beta})$ and $S(f|\Gamma_2) < \infty$ such that

$$\frac{S(Q[f]|\Gamma_2) + S(\bar{P}_1[f]|\Gamma_2)}{2} \geq (1-\epsilon)S(f|\Gamma_2)$$

Proof.- The motivation behind the proof is the following idea. This statement is opposite Han's inequality (2.20) in nature. There, the conclusion is $S(\bar{P}_1 h) + S(\bar{P}_2 h) \leq 1S(h)$. One can attribute the success of Han's inequality to the "orthogonality" of the directions in which the thermostats P_1 and P_2 act. Q and $\bar{P}_1$ should act "tangentially" so that the constant $(1-\epsilon)$ is obtained.

Let h be the characteristic function of the rectangle $[-a, a] \times [R-a, R+a]$, normalized to have $\Gamma_{2,\beta}$-integral 1. $h(v_1, v_2) = \frac{\mathbf{1}_{[-a,a]}(v_1)\mathbf{1}_{[R-a,R+a]}(v_2)}{Z_0^a\, Z_R^a}$.

$$\text{Here} \qquad Z_x^y = \int_{x-y}^{x+y} \Gamma_{2,\beta}(v_1, v_2)\, dv_1\, dv_2.$$

Then $S(h) = -\ln(Z_0^a) - \ln(Z_R^a)$. $\bar{P}_1[h](v_1, v_2) = \frac{\mathbf{1}_{[R-a,R+a]}(v_2)}{Z_R^a}$. Thus $S(\bar{P}_1 h) = -\ln(Z_R^a)$. Qh is supported in the annulus $A := R - a \leq |v| \leq \sqrt{(R+a)^2 + a^2}$. So

$$S(Qh) \geq S(|A|^{-1}\mathbf{1}_A(v_1, v_2))$$

with $|A| = \int_A \Gamma(v_1, v_2)\, dv_1\, dv_2$. This can be seen from

$$\begin{aligned}
S(Qh) &= \int_{\mathbb{R}^2} [Qh] \ln[Qh] \Gamma_2(v_1, v_2)\, dv = \int_A [Qh] \ln[Qh] \Gamma_2(v_1, v_2)\, dv \\
&= |A| \int_A [Qh] \ln[Qh] |A|^{-1} \Gamma_2(v_1, v_2)\, dv \\
&\geq |A| \left(\int_A [Qh] |A|^{-1} \Gamma_2(v_1, v_2) \right) \ln \left(\int_A [Qh] |A|^{-1} \Gamma_2(v_1, v_2) \right) dv \\
&= |A| (|A|^{-1}) \ln(|A|^{-1}) = -\ln(|A|) = S(|A|^{-1}\mathbf{1}_A)
\end{aligned}$$

Here, Jensen's inequality was used.

It follows that

$$\frac{S(Q[f]\|\Gamma_2) + S(\bar{P}_1[f]\|\Gamma_2)}{S(f|\Gamma_2)} \geq \frac{S(|A|^{-1}\mathbf{1}_A) + S(\bar{P}_1 h)}{S(h)} = \frac{-\ln(|A|) - \ln(Z_R^a)}{-\ln(Z_0^a) - \ln(Z_R^a)}.$$

$$\limsup_{R\to\infty} \frac{S(Q[f]\|\Gamma_2) + S(\bar{P}_1[f]\|\Gamma_2)}{S(f|\Gamma_2)} \quad \geq \quad \limsup_{R\to\infty} \frac{-\ln(|A|) - \ln(Z_R^a)}{-\ln(Z_0^a) - \ln(Z_R^a)}.$$

Because $|A| = 2\pi \frac{\beta}{2\pi} \int_{R_1}^{R_2} \exp(-\frac{\beta}{2} r^2) r\, dr = \exp(-\frac{\beta}{2}(R-a)^2) - \exp(-\frac{\beta}{2}((R+a)^2 + a^2))$, we have

$$|A| = \exp(-\frac{\beta}{2}(R-a)^2) \left(1 - \exp(-\frac{\beta}{2}[(R+a)^2 + a^2 - (R-a)^2]) \right)$$

which lies in the interval $\left[\frac{1}{2}\exp(-\frac{\beta}{2}(R-a)^2), \exp(-\frac{\beta}{2}(R-a)^2)\right]$ for large R.
Thus, $-\ln|A| = \frac{\beta}{2}(R-a)^2 + O(1)$ as $R \to \infty$.

Similarly, we have Z_R^a is between $[\frac{\beta}{2\sqrt{2\pi}}e^{-\frac{\beta}{2}(R-a)^2}, \frac{\beta}{2\sqrt{2\pi}}e^{-\frac{\beta}{2}(R-a)^2}]$ when R is large enough. It follows that , $-\ln(Z_R^a) = \frac{\beta}{2}(R-a)^2 + O(\ln(R))$. While Z_0^a does not depend on R.

Therefore $\limsup_{R\to\infty} \frac{S(Q[f]||\Gamma_2)+S(\bar{P}_1[f]||\Gamma_2)}{S(f|\Gamma_2)} \geq \limsup_{R\to\infty} \frac{\beta(R-a)^2+O(\ln(R))}{\frac{\beta}{2}(R-a)^2+O(1)} = 2.$ $\square$

Appendix B

Let $f_0(v,w) = l_0(v)\Gamma_{\mathcal{N}}(w) = h_0(v)\Gamma_{n+\mathcal{N}}(v,w)$ be a probability density in $L^1(\mathbb{R}^{n+\mathcal{N}})$ with finite second moment. Let f_∞ be the radial projection of f_0. Then clearly $f_\infty = h_\infty \Gamma_{n+\mathcal{N}}$ where h_∞ is the radial projection of h. In this appendix, we show that if f is centered at the origin then

$$d_2(f_\infty, \Gamma_{n+\mathcal{N},\beta}) \leq \frac{n}{n+\mathcal{N}} d_2(f_0, \Gamma_{n,\beta}), \tag{6.3}$$

and if $h_0 \in L^2(\Gamma_{n+\mathcal{N}})$ then

$$||h_\infty - 1||_{L^2(\Gamma_{n+\mathcal{N}})} \leq \sqrt{\frac{n}{\mathcal{N}-2}} ||h_0 - 1||_{L^2(\Gamma_n)} \tag{6.4}$$

Proof.- Both inequalities are based on the fact that if ϕ is a function of n variables then R_ϕ, its radial projection on $n+\mathcal{N}$ can be computed using Cartesian parametric coordinates after projecting $\{|v|^2+|w|^2=r^2\}$ onto the plane $w_{\mathcal{N}}=0$. Let $\tilde{w} = (w_1, \ldots, w_{\mathcal{N}-1})$.

$$\begin{aligned}
R_\phi(r) =& \frac{2}{|S^{n+\mathcal{N}-1}|\, r^{n+\mathcal{N}-1}} \int_{\{|v|^2+|\tilde{w}|^2 \leq r^2\}} \phi(v) \frac{r}{|w_{\mathcal{N}}|} dv\, d\tilde{w} \\
=& \frac{2}{|S^{n+\mathcal{N}-1}|\, r^{n+\mathcal{N}-1}} \int_{|v|\leq r} \phi(v) \frac{r}{\sqrt{r^2-v^2}} \int_{\tilde{w}^2 \leq r^2 - v^2} \sqrt{\frac{r^2-v^2}{r^2-v^2-\tilde{w}^2}} d\tilde{w}\, dv \\
=& \frac{2\,|S^{\mathcal{N}-2}|}{|S^{n+\mathcal{N}-1}|\, r^{n+\mathcal{N}-1}} \int_{|v|\leq r} \phi(v) \frac{r}{\sqrt{r^2-|v|^2}} (r^2-|v|^2)^{\frac{\mathcal{N}-1}{2}} \int_0^1 \frac{t^{\mathcal{N}-2}}{\sqrt{1-t^2}}\, dt \\
=& \frac{2\,|S^{\mathcal{N}-2}|}{|S^{n+\mathcal{N}-1}|\, r^{n}} \int_{|v|\leq r} \phi(v) \left(1 - \frac{|v|^2}{r^2}\right)^{\frac{\mathcal{N}-2}{2}} dv \quad \frac{\sqrt{\pi}\Gamma[\frac{\mathcal{N}-1}{2}]}{2\Gamma[\frac{\mathcal{N}}{2}]} \\
=& \frac{|S^{\mathcal{N}-1}|}{|S^{n+\mathcal{N}-1}|\; r^{n}} \int_{|v|\leq r} \phi(v) \left(1 - \frac{|v|^2}{r^2}\right)^{\frac{\mathcal{N}-2}{2}} dv
\end{aligned} \tag{6.5}$$

Proof (of (6.3)).- We have

$$\begin{aligned}
d_2(f_\infty, \Gamma_{n+\mathcal{N}}) =& \sup_{r\neq 0} \int_{S^{n+\mathcal{N}-1}(r)} \frac{[\widehat{l_0}(\vec{\xi}) - \Gamma_n(\vec{\xi})]}{r^2} \Gamma_{\mathcal{N}}(\vec{\eta}) d\sigma_r(\vec{\xi}, \vec{\eta}) \\
\leq & \left(\sup_{r\neq 0} \int_{S^{n+\mathcal{N}-1}(r)} \frac{|\vec{\xi}|^2}{r^2} \Gamma_{\mathcal{N}}(\vec{\eta}) d\sigma_r(\vec{\xi}, \vec{\eta}) \right) d_2(l_0, \Gamma_n) \\
\leq & \left(\sup_{r\neq 0} \frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right| r^n} \int_{|\vec{\xi}|\leq r} \frac{|\vec{\xi}|^2}{r^2} \left(1 - \frac{|\vec{\xi}|^2}{r^2}\right)^{\frac{\mathcal{N}-2}{2}} d\vec{\xi} \right) d_2(l_0, \Gamma_n) \\
=& \frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right|} \left(\sup_{r>0} \int_{|\vec{\xi}|\leq r} \frac{|\vec{\xi}|^2}{r^2} \left(1 - \frac{|\vec{\xi}|^2}{r^2}\right)^{\frac{\mathcal{N}-2}{2}} \frac{d\vec{\xi}}{r^n} \right)) d_2(l_0, \Gamma_n) \\
\leq & \frac{\left|S^{\mathcal{N}-1}\right| \left|S^{n-1}\right|}{\left|S^{n+\mathcal{N}-1}\right|} \int_0^1 \rho^{n+1} (1-\rho^2)^{\frac{\mathcal{N}-1}{2}} \, d\rho \quad d_2(l_0, \Gamma_n)
\end{aligned}$$

which using $s = \rho^2$ and the formula for the β function, the coefficient of $d_2(l_0, \Gamma_n)$ becomes:

$$\frac{1}{2} \frac{2\pi^{\frac{n}{2}} 2\pi^{\frac{\mathcal{N}}{2}} \Gamma\left(\frac{n+\mathcal{N}}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{\mathcal{N}}{2}\right) 2\pi^{\frac{n+\mathcal{N}}{2}}} \frac{\Gamma\left(\frac{n}{2}+1\right) \Gamma\left(\frac{\mathcal{N}}{2}\right)}{\Gamma\left(\frac{n+\mathcal{N}}{2}+1\right)} = \frac{n}{n+\mathcal{N}}.$$

□

Proof (of (6.4)).- By (6.5) we have

$$h_\infty(r) = \frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right| r^n} \int_{\mathbb{R}^n} h_0(v) \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} dv$$

where $x_+ = x$ if $x \geq 0$ and $x_+ = 0$ otherwise.

In particular,

$$\frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right| r^n} \int_{\mathbb{R}^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} dv = 1$$

Using $\int \Gamma_{\mathcal{N}}(v) h_0(v) dv = 1$, we may write

$$\begin{aligned}
h_\infty(r) - 1 =& \int_{\mathbb{R}^n} \left[\frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right| r^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} - \Gamma_{\mathcal{N}}(v) \right] (h_0(v) - 1) dv \\
=& \int_{\mathbb{R}^n} \left[\frac{\left|S^{\mathcal{N}-1}\right|}{\left|S^{n+\mathcal{N}-1}\right| r^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} - 1 \right] \sqrt{\Gamma_{\mathcal{N}}} (h_0(v) - 1) \sqrt{\Gamma_{\mathcal{N}}} dv
\end{aligned}$$

and using Cauchy-Schwarz's inequality we find that

$$|h_\infty(r) - 1|^2 \leq \int_{\mathbb{R}^n} \Gamma_{\mathcal{N}}(v)(h_0(v)-1)^2 dv \int_{\mathbb{R}^n} \left[\frac{|S^{\mathcal{N}-1}|}{|S^{n+\mathcal{N}-1}|r^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} - 1 \right]^2 \Gamma_{\mathcal{N}} dv \, .$$

Integrating both sides on $\mathbb{R}^{n+\mathcal{N}}$ with respect to the Lebesgue measure, we get

$$\|h_\infty - 1\|^2 = |S^{n+\mathcal{N}-1}| \int r^{n+\mathcal{N}-1} e^{-\pi r^2} |h_\infty(r) - 1| dr \leq C \|h_0 - 1\|_2^2$$

where

$$C = |S^{n+\mathcal{N}-1}| \int_0^\infty dr\, r^{n+\mathcal{N}-1} e^{-\pi r^2} \int_{\mathbb{R}^n} \left[\frac{|S^{\mathcal{N}-1}|}{|S^{n+\mathcal{N}-1}|r^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} e^{\pi|v|^2/2} - e^{-\pi|\vec{v}|^2/2} \right]^2 dv$$

The right-hand side can be computed exactly by expanding the square and evaluating the three terms using:

$$\begin{aligned}
\int_0^\infty dr r^{n+\mathcal{N}-1} e^{-\pi r^2} \int_{\mathbb{R}^n} \frac{|S^{\mathcal{N}-1}|^2}{|S^{n+\mathcal{N}-1}|r^{2n}} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)} e^{\pi|v|^2} dv = \frac{\Gamma(\frac{n+\mathcal{N}}{2})}{\Gamma(\frac{\mathcal{N}}{2})\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{\mathcal{N}-2}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{n+\mathcal{N}-2}{2})} = \frac{n+\mathcal{N}-2}{\mathcal{N}-2} \, , \\
\int_0^\infty dr\, r^{n+\mathcal{N}-1} e^{-\pi r^2} \int_{\mathbb{R}^n} \frac{|S^{\mathcal{N}-\infty}|}{r^n} \left(1 - \frac{|v|^2}{r^2}\right)_+^{(\mathcal{N}-2)/2} dv = 1 \, , \\
|S^{n+\mathcal{N}-1}| \int_0^\infty dr r^{n+\mathcal{N}-1} e^{-\pi r^2} \int_{\mathbb{R}^n} e^{-\pi|v|^2} dv = 1 \, .
\end{aligned} \tag{6.6}$$

We thus get

$$C = \frac{n}{\mathcal{N}-2} \, .$$

Appendix C

In this appendix (taken from Appendix B in [3]), we show that there exists an initial state u_0 for which we have

$$\|(Q_I - Q_M)u_0\|_2 \geq C \frac{n}{\sqrt{\mathcal{N}}} \|u_0\|_2.$$

thus saturating the bound in Lemma 6.

We first observe that, by a similar analysis as Lemma 6, we get

$$\left\|\sum_{i=1}^{n}\left(\frac{1}{\mathcal{N}}\sum_{j=1}^{\mathcal{N}}Q^I_{i,j}u - M_iu\right)\right\|_2^2 = \frac{n}{\mathcal{N}}\left(\langle M_1u,u\rangle - \langle M_1u,M_1u\rangle\right)+\frac{n(n-1)}{\mathcal{N}}\left(\langle Q^I_{1,1}u,Q^I_{2,1}u\rangle - \langle M_1u,M_2u\rangle\right)$$

Thus it suffices to find a symmetric initial states such that $\langle Q^I_{1,1}u,Q^I_{2,1}u\rangle - \langle M_1u,M_2u\rangle = O(1)$ in n and $\mathcal{N}$. To this end we set

$$u_{n,P}(v) = \sum_{p_1+p_2+\cdots+p_n=P}\prod_{i=1}^{n} H_{2p_i}(v_i)$$

Observe that

$$u_{n,P}(v) = \sum_{p_1+p_2\le P} H_{2p_1}(v_1)H_{2p_2}(v_2)u_{M-2,P-p_1-p_2}(v^{1,2})$$

Clearly we have

$$Q^I_{1,1}H_{2p_1}(v_1) = \widetilde{H}_{2p_1}(v_1,w_1)$$

where $\bar{u}(v_1,v_2) = H_4(v_1)+H_2(v_1)H_2(v_2)+H_4(v_2)$. Thus

$$Q^I_{1,1}u_{n,P}(v) = \sum_{p_1+p_2\le P}\widetilde{H}_{2p_1}(v_1,w_1)H_{2p_2}(v_2)u_{n-2,P-p_1-p_2}(v^{1,2})\ .$$

$$\langle Q^I_{1,1}H_{2p_1}(v_1)H_{2p_2}(v_2),Q^I_{1,1}H_{2p_1}(v_1)H_{2p_2}(v_2)\rangle - \langle M_1H_{2p_1}(v_1)H_{2p_2}(v_2),M_2H_{2p_1}(v_1)H_{2p_2}(v_2)\rangle \ge 0$$

so that we have

$$\langle Q^I_{1,1}u_{n,P},Q^I_{2,1}u_{n,P}\rangle - \langle M_1u_{n,P},M_2u_{n,P}\rangle \ge \left(\langle Q^I_{1,1}\bar{u},Q^I_{2,1}\bar{u}\rangle - \langle M_1\bar{u},M_2\bar{u}\rangle\right)\|u_{P-2,n-2}\|_2$$

Observe now that $\|u_{P,n}\|_2 = \binom{n+P}{P-1}$ while $\langle Q^I_{1,1}\bar{u},Q^I_{2,1}\bar{u}\rangle - \langle M_1\bar{u},M_2\bar{u}\rangle = \frac{11}{8}$ so that

$$\langle Q^I_{1,1}u_{n,P},Q^I_{2,1}u_{n,P}\rangle - \langle M_1u_{n,P},M_2u_{n,P}\rangle \ge \frac{11}{8}\frac{(P-1)(P-2)(n+1)n}{(n+P)(n+P-1)(n+P-2)(n+P-3)}\|u_{n,P}\|_2.$$

By choosing $P=n$ we get

$$\langle Q^I_{1,1}u_{n,n},Q^I_{2,1}u_{n,n}\rangle - \langle M_1u_{n,n},M_2u_{n,n}\rangle \ge C\|u_{n,n}\|_2$$

with $C = 3/128$.

We can thus consider an initial state given by

$$h_0(v) = 1 + a u_{n,n}(v).$$

Observe that $u_{n,n}$ is an even polynomial in all its variables with positive coefficients for the terms of maximal degree. Thus $\inf_{\mathbb{R}^n} u_{n,n}(v) > -\infty$ and choosing a small enough we get $h_0 \geq 0$. $\square$

Appendix D

This appendix (taken from Appendix C in [3]) shows that if $K > 0$ and let $\mathcal{N}$ be large enough, then there is a function H that is $C^4(\mathbb{R})$, even, and $H(0) = 0$ such that

$$\mathcal{D}_{\mathcal{N}}(H, a) > K\mathcal{D}_1(H, a).$$

for all $a \neq 0$.

Proof.- Consider the function $H_r(x) = x^4 \exp(-rx^2)$ parametrized by r. We have

$$D_1(a) = \sup_{x \neq 0} \frac{|H_r(x)|}{a^2 + x^2} = \sup_{u > 0} \frac{u^2 e^{-ru}}{a^2 + u}.$$

When $r \geq \frac{1}{a^2}$, This supremum occurs at $u_*(r) \leq \frac{2a^2}{r}$ which is positive. For large r, $u_*(r) \to 0$. Thus

$$D_{\mathcal{N}}(a) \geq \frac{\sum_{i=1}^{\mathcal{N}} H_r(\sqrt{u_*(r)}) e^{-\pi(\mathcal{N}-1)u_*(r))}}{a^2 + \mathcal{N} u_*(r)} = \frac{\mathcal{N} u_*(r)^2 e^{-r(\mathcal{N}-1)u_*(r)}}{a^2 + \mathcal{N} u_*(r)}$$

Hence

$$\liminf_{r \to \infty} \frac{D_{\mathcal{N}}(a)}{D_1(a)} \geq \liminf_{r \to \infty} \frac{a^2 + u_*(r)}{\frac{a^2}{\mathcal{N}} + u_*(r)} \exp(-\pi(\mathcal{N} - 1)u_*) = \mathcal{N}.$$

Note also that this bound is optimal since for any H and a we have

$$\mathcal{D}_{\mathcal{N}}(H, a) \leq \sup_{\eta} \frac{\sum_{i=1}^{\mathcal{N}} \mathcal{D}_1(H, a)(a^2 + \eta^2)}{a^2 + \mathcal{N}\eta^2} \leq \mathcal{N}\mathcal{D}_1(H, a). \tag{6.7}$$

$\square$

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